E7B.3(a) The normalized wavefunction has the form $\psi(\phi) = Ne^{i\phi}$ where N is the normalization constant.

$$\int_0^{2\pi} \psi^* \psi \, d\varphi = 1 \text{ [7B.4c]}$$

$$N^2 \int_0^{2\pi} e^{-i\phi} e^{i\phi} \, d\phi = N^2 \int_0^{2\pi} d\phi = 2\pi N^2 = 1$$

$$\boxed{N = \left(\frac{1}{2\pi}\right)^{1/2}}$$

B.5 The normalized wavefunction is ✓

$$\psi(r) = \left(\frac{1}{\pi a_0^3}\right)^{1/2} e^{-r/a_0} \quad \text{and} \quad \psi(r)^2 = \frac{1}{\pi a_0^3} e^{2r/a_0} \quad \text{where } a_0 = 53 \text{ pm}$$

The probability P that the particle will be found in the region is the integral summation of all the infinitesimal probabilities of finding the particle within the region: $P = \int_{\text{region}} \psi(r)^2 \, d\tau$. In this problem we are to integrate over the region of a sphere for which the radius (b = 1.0 pm = a_0/53) is a very small fraction of the characteristic wavefunction distance a_0. Consequently, we may assume that ψ does not vary much within the sphere. Then, the probability is given by

$$P = \int_{\text{region}} \psi(r)^2 \, d\tau \sim \psi(a)^2 \int_{\text{region}} d\tau$$

where a is the position of the centre of the sphere

$$\sim \left(\frac{4\pi b^3}{3}\right)\left(\frac{1}{\pi a_0^3}\right) e^{-2a/a_0}$$

$$\sim \left(\frac{4}{3}\right)\left(\frac{b}{a_0}\right)^3 e^{\;2a/a_0}$$

(a) $b = 1.0\ \text{pm} = a_0/53$ is the radius of a sphere that is centred on the origin ($a = 0$).

$$P(0,b) = \frac{4}{3\times(53)^3} e^{-0} = \boxed{9.0\times10^{-6}}$$

(b) $b = 1.0\ \text{pm} = a_0/53$ is the radius of a sphere that is centred at $a = a_0$.

$$P = \left(\frac{4}{3\times(53)^3}\right) e^{-2} = \boxed{1.2\times10^{-6}}$$

E7C.2(a) Let f and g be functions of x and examine the integral $\int_{-\infty}^{\infty} f^*\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right)g\,dx$.

Integrate successively by parts and use the fact that these functions must be well behaved at the boundaries (i.e. they equal zero at infinity in either direction).

$$\int_{-\infty}^{\infty} f^*\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right)g\,dx=\left(-\frac{\hbar^2}{2m}\right)\int_{-\infty}^{\infty} f^*\left(\frac{d^2}{dx^2}\right)g\,dx=\left(-\frac{\hbar^2}{2m}\right)\int_{-\infty}^{\infty} f^*\left(\frac{d}{dx}\right)\frac{dg}{dx}\,dx$$

$$=\left(-\frac{\hbar^2}{2m}\right)\times\left\{\left[f^*\frac{dg}{dx}\right]_{-\infty}^{\infty}-\int_{-\infty}^{\infty}\frac{df^*}{dx}\times\frac{dg}{dx}\,dx\right\}$$

$$=-\left(-\frac{\hbar^2}{2m}\right)\times\left\{\int_{-\infty}^{\infty}\frac{df^*}{dx}\times\frac{dg}{dx}\,dx\right\}$$

$$=-\left(-\frac{\hbar^2}{2m}\right)\times\left\{\left[g\frac{df^*}{dx}\right]_{-\infty}^{\infty}-\int_{-\infty}^{\infty}g\times\frac{d^2f^*}{dx^2}\,dx\right\}$$

$$=\left(-\frac{\hbar^2}{2m}\right)\times\left\{\int_{-\infty}^{\infty}g\times\frac{d^2f^*}{dx^2}\,dx\right\}$$

$$=\int_{-\infty}^{\infty}g\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right)f^*\,dx=\left\{\int_{-\infty}^{\infty}g^*\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right)f\,dx\right\}^*$$

Thus, $\int_{-\infty}^{\infty} f^*\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right)g\,dx=\left\{\int_{-\infty}^{\infty}g^*\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right)f\,dx\right\}^*$

This is exactly the criteria that a hermitian operator must satisfy [7C.7] so we conclude that the kinetic energy operator is hermitian.

E7C.3(a) ψ_i and ψ_j are orthogonal if $\int \psi_i^* \psi_j \, d\tau = 0$ [7C.8]. Where $n \neq m$ and both n and m are integers, explicit integration gives

$$\int_0^L \sin(n\pi x/L) \times \sin(m\pi x/L)\,dx$$

$$= \left[\frac{\sin(\pi(n-m)x/L)}{2\pi(n-m)/L} - \frac{\sin(\pi(n+m)x/L)}{2\pi(n+m)/L}\right]_{x=0}^{x=L}$$

$$= \frac{\sin(\pi(n-m))}{2\pi(n-m)/L} - \frac{\sin(\pi(n+m))}{2\pi(n+m)/L} - \left\{\frac{\sin(0)}{2\pi(n-m)/L} - \frac{\sin(0)}{2\pi(n+m)/L}\right\}$$

$= 0$ because the sine of an integer multiple of π equals zero.

Thus, the functions $\sin(n\pi x/L)$ and $\sin(m\pi x/L)$ are orthogonal in the region $0 \le x \le L$.

Alternatively, successively integrate by parts.

$$\int_0^L \sin(n\pi x/L) \times \sin(m\pi x/L)\,dx$$

$$= \left[\sin(n\pi x/L) \times \left(\frac{-\cos(m\pi x/L)}{m\pi/L}\right)\right]_{x=0}^{x=L}$$

$$- \int_0^L (n\pi/L)(\cos(n\pi x/L)) \times \left(\frac{\cos(m\pi x/L)}{m\pi/L}\right) dx \quad \text{[integration by parts]}$$

$$= \left(\frac{n}{m}\right)\int_0^L \cos(n\pi x/L) \times \cos(m\pi x/L)\,dx \quad [\text{use } \sin(n\pi) = 0 \text{ for multiples of } \pi]$$

$$= \left(\frac{n}{m}\right) \times \left\{\left[\cos(n\pi x/L) \times \left(\frac{\sin(m\pi x/L)}{m\pi/L}\right)\right]_{x=0}^{x=L} - \int_0^L (n\pi/L)(-\sin(n\pi x/L)) \times \left(\frac{\sin(m\pi x/L)}{m\pi/L}\right) dx\right\}$$

[integration by parts]

$$= \left(\frac{n}{m}\right)^2 \times \left\{\int_0^L \sin(n\pi x/L) \times \sin(m\pi x/L)\,dx\right\} \quad [\text{use } \sin(m\pi) = 0 \text{ for multiples of } \pi]$$

Thus, $$\left(1 - \left(\frac{n}{m}\right)^2\right) \times \left\{\int_0^L \sin(n\pi x/L) \times \sin(m\pi x/L)\,dx\right\} = 0$$

and we conclude that the integral necessarily equals zero when $n \neq m$.

E7C.4(a) ψ_i and ψ_j are orthogonal if $\int \psi_i^* \psi_j \, d\tau = 0$ [7C.8].

$$\int_0^{2\pi} \left(e^{i\phi}\right)^* \times e^{2i\phi}\,d\varphi = \int_0^{2\pi} e^{-i\phi} \times e^{2i\phi}\,d\varphi = \int_0^{2\pi} e^{i\phi}\,d\phi = \frac{1}{i} e^{i\phi} \Big|_{\phi=0}^{\phi=2\pi}$$

$$= \frac{1}{i}\left(e^{2\pi i} - e^0\right) = \frac{1}{i}\left(e^{2\pi i} - 1\right) = \frac{1}{i}\left(\cos(2\pi) - i\sin(2\pi) - 1\right) = \frac{1}{i}(1 - 0 - 1) = 0$$

(The Euler identity $e^{ai} = \cos(a) - i\sin(a)$ has been used in the math manipulations.)
Thus, the functions $e^{i\phi}$ and $e^{2i\phi}$ are orthogonal in the region $0 \le \phi \le 2\pi$

E7C.6(a) The normalized form of this

$$\psi(x) = \left(\frac{2}{L}\right)^{1/2} \sin(\pi x / L)$$

$$\frac{d\psi}{dx} = \left(\frac{2}{L}\right)^{1/2} \left(\frac{\pi}{L}\right) \cos(\pi x / L)$$

The expectation value of the electron momentum is:

$$\langle p_x \rangle = \int_0^L \psi^* \hat{p}_x \psi \, dx \; [7C.11] = \int_0^L \psi^* \left(\frac{\hbar}{i}\frac{d}{dx}\right) \psi \, dx = \left(\frac{\hbar}{i}\right) \int_0^L \psi^* \left(\frac{d\psi}{dx}\right) dx$$

$$= \left(\frac{\hbar}{i}\right)\left(\frac{2}{L}\right)^{1/2}\left(\frac{\pi}{L}\right) \int_0^L \psi^* \cos\left(\frac{\pi x}{L}\right) dx = \left(\frac{\hbar}{i}\right)\left(\frac{2}{L}\right)\left(\frac{\pi}{L}\right) \int_0^L \sin\left(\frac{\pi x}{L}\right) \cos\left(\frac{\pi x}{L}\right) dx$$

$$= \left(\frac{h}{iL^2}\right) \int_0^L \sin\left(\frac{\pi x}{L}\right) \cos\left(\frac{\pi x}{L}\right) dx$$

$$= \left(\frac{h}{iL^2}\right) \times \left[\frac{\sin^2\left(\frac{\pi x}{L}\right)}{2\pi / L}\right]_{x=0}^{x=L} = \left(\frac{h}{iL^2}\right) \times \left(\frac{\sin^2(\pi)}{2\pi / L} - \frac{\sin^2(0)}{2\pi / L}\right) = \left(\frac{h}{iL^2}\right) \times (0+0) = \boxed{0}$$

E7C.8(a) The desired uncertainty in the proton momentum is

$$\Delta p = 1.00\times10^{-4}\ p = 1.00\times10^{-4}\ m_p v$$
$$= (1.00\times10^{-4})\times(1.673\times10^{-27}\ \text{kg})\times(0.45\times10^{6}\ \text{m s}^{-1}) = 7.5\overline{3}\times10^{-26}\ \text{kg m s}^{-1}$$

Thus, the minimum uncertainty in position must be

$$\Delta x = \frac{\hbar}{2\Delta p}\ [7\text{C}.13\text{a}] = \frac{1.055\times10^{-34}\ \text{J s}}{2\times(7.53\times10^{-26}\ \text{kg m s}^{-1})} = \boxed{700\ \text{pm}}$$

E7C.9(a) The quantity $[\hat{\Omega}_1,\hat{\Omega}_2] = \hat{\Omega}_1\hat{\Omega}_2 - \hat{\Omega}_2\hat{\Omega}_1$ [7C.15] is referred to as the commutator of the operators Ω_1 and Ω_2. In obtaining the commutator it is necessary to realize that the operators operate on functions; thus, we find expressions for $[\hat{\Omega}_1,\hat{\Omega}_2]\psi(x) = \hat{\Omega}_1\hat{\Omega}_2\psi(x) - \hat{\Omega}_2\hat{\Omega}_1\psi(x)$.

(a) $$\left[\frac{\text{d}}{\text{d}x},\frac{1}{x}\right]\psi = \frac{\text{d}}{\text{d}x}\times\left(\frac{1}{x}\psi\right) - \frac{1}{x}\frac{\text{d}}{\text{d}x}\psi = \left(-\frac{1}{x^2}\right)\psi + \frac{1}{x}\frac{\text{d}}{\text{d}x}\psi - \frac{1}{x}\frac{\text{d}}{\text{d}x}\psi = \left(-\frac{1}{x^2}\right)\psi$$

Thus, $$\left[\frac{\text{d}}{\text{d}x},\frac{1}{x}\right] = \boxed{-\frac{1}{x^2}}$$

(b) $$\left[\frac{\text{d}}{\text{d}x},x^2\right]\psi = \frac{\text{d}}{\text{d}x}\times(x^2\psi) - x^2\frac{\text{d}}{\text{d}x}\psi = (2x)\psi + x^2\frac{\text{d}}{\text{d}x}\psi - x^2\frac{\text{d}}{\text{d}x}\psi = (2x)\psi$$

Thus, $$\left[\frac{\text{d}}{\text{d}x},x^2\right] = \boxed{2x}$$

P7C.1 Time-independent Schrödinger equation of a single particle in one dimension:

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+\hat{V}\right)\psi = E\psi \quad [7B.1]$$

(a) $$\left(-\frac{\hbar^2}{2m_e}\frac{d^2}{dx^2}-\frac{e^2}{4\pi\varepsilon_0 x}\right)\psi = E\psi$$

(b) $$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right)\psi = E\psi$$

(c) $F = c$ (a constant) implies that $V = -cx$ because $F = -dV/dx$.

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}-cx\right)\psi = E\psi$$

P7C.3 Form $\hat{\Omega}f$ in each case. If the result is ωf where ω is a constant, then f is an eigenfunction of the operator $\hat{\Omega}$ and ω is the eigenvalue [7C.2a, b, and c]. We check whether f is an eigenfunction of the $\Omega = d/dx$ operator.

(a) $$\frac{d}{dx}f = \frac{d}{dx}e^{ikx} = ike^{ikx} = ikf$$

Yes, the function is an eigenfunction with the eigenvalue ik.

(b) $$\frac{d}{dx}f = \frac{d}{dx}k = 0 = 0\times f$$

Yes, the function is an eigenfunction with the eigenvalue 0.

(c) $$\frac{d}{dx}f = \frac{d}{dx}kx = k$$

No, the function is not an eigenfunction because $k \neq \text{constant} \times f$.

(e) $$\frac{d}{dx}f = \frac{d}{dx}e^{-ax} = -2ax\,e^{-ax} = -2axf$$

No, the function is not an eigenfunction because $-2ax \neq$ constant.

P7C.7 (a) The function e^{+ikx} is an eigenfunction of the linear momentum operator, $\hat{p}_x = \frac{\hbar}{i}\frac{d}{dx}$ [7C.3]. It has the eigenvalues $+k\hbar$:

$$\hat{p}e^{+ikx} = \frac{\hbar}{i}\frac{d}{dx}e^{+ikx} = \left(\frac{\hbar}{i}\right)\times(ik)e^{+ikx} = +k\hbar e^{+ikx}$$

Consequently, the particle has the linear momentum $+k\hbar$.

(b) The wavefunction $\psi = N\cos kx$ is not an eigenfunction of the linear momentum operator so we find the expectation value for linear momentum with eqn 7C.11.

$$\langle p_x\rangle = \int_{-\infty}^{\infty}\psi^*\hat{p}_x\psi dx \ [7C.11] = N^2\int_{-\infty}^{\infty}\cos kx\left(\frac{\hbar}{i}\frac{d}{dx}\right)\cos kx dx$$

$$= N^2\left(\frac{\hbar}{i}\right)\int_{-\infty}^{\infty}\cos kx\left(\frac{d}{dx}\right)\cos kx dx = -kN^2\left(\frac{\hbar}{i}\right)\int_{-\infty}^{\infty}\cos kx\times\sin kx dx$$

$$= -kN^2\left(\frac{\hbar}{i}\right)\lim_{\chi\to\infty}\left[\frac{\sin^2 kx}{2k}\right]_{x=-\chi}^{x=\chi} \quad \text{[standard integral]}$$

$$= -kN^2\left(\frac{\hbar}{i}\right)\lim_{\chi\to\infty}\left[\frac{\sin^2(k\chi)}{2k} - \frac{\sin^2(-k\chi)}{2k}\right]$$

$$= -kN^2\left(\frac{\hbar}{i}\right)\lim_{\chi\to\infty}\left[\frac{\sin^2(k\chi)}{2k} - \frac{\sin^2(k\chi)}{2k}\right] = \boxed{0}$$

(c) $\psi = Ne^{-ax^2}$

$$\frac{d}{dx}\psi = N\frac{d}{dx}e^{-ax^2} = -2aNxe^{-ax^2} = -2ax\psi$$

The wavefunction is not an eigenfunction of the linear momentum operator so we find the expectation value for linear momentum with eqn 7C.11.

$$\langle p_x\rangle = \int_{-\infty}^{\infty}\psi^*\hat{p}_x\psi dx \ [7C.11] = N^2\int_{-\infty}^{\infty}e^{-ax^2}\left(\frac{\hbar}{i}\frac{d}{dx}\right)e^{-ax^2} dx$$

$$= -2aN^2\left(\frac{\hbar}{i}\right)\int_{-\infty}^{\infty}xe^{-2ax^2} dx$$

The integrand of the above integral is an odd function so, when it is integrated around its centre of symmetry at $x = 0$, the integral equals zero. Thus, $\langle p_x \rangle = \boxed{0}$.

P7C.13 For the hermitian operator Ω:

$$\langle \Omega^2 \rangle = \int \psi^* \hat{\Omega}^2 \psi d\tau = \int \psi^* \hat{\Omega}\left(\hat{\Omega}\psi\right) d\tau = \left\{ \int \left(\hat{\Omega}\psi\right)^* \hat{\Omega}\psi d\tau \right\}^* \text{ [7C.7]}$$

The integrand on the far right is a function times its complex conjugate, which must always be a real, positive number. When this type of integrand is integrated over real space, the result is always a real, positive number. Thus, the expectation value of the square of a hermitian operator is always positive.