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Two oscillators in a dissipative bath

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Abstract

Two forms of dissipation are applied to a system of two coupled harmonic oscillators. Resulting quantum master equations are studied with the help of quantum characteristic functions. The effects of dissipation on quantum entanglement are investigated in the view of recent interest in separability criteria for density matrices. Decoherence-free subspaces are constructed for a Lindlad dissipation mechanism.

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1. Introduction

In this paper we extend a previous treatment [1] of one primary oscillator in a dissipative bath of various types to a system of two coupled primary oscillators. The purpose here is several fold. We are interested in dynamics of quantum dissipation in systems which can be described by two coupled harmonic oscillators. Methods employed in Ref. [1] are naturally extendable to multi-oscillator systems, and are expected to capture to the full extent the dissipative dynamics of various physical and chemical natures [2,3]. On the other hand, in the emerging literature [4–10] of quantum information and communication, a tremendous amount of interest has risen in bipartite canonical systems of continuous variables which in that field are often synonymous to a pair of harmonic oscillators. Entanglement (or inseparability) and its quantification are some of the central issues of fundamental importance to quantum computing, quantum communication, and quantum mechanics itself.

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Although entanglement properties of bipartite systems have been studied in the presence of dissipative baths, considerations of the baths have not been extensive [11–13]. In this paper we approach the problem starting from constructing physically meaningful baths. As an extension of our previous work [1] we adopt a form of dissipative baths first proposed by Agarwal [14]. Three important attributes are desired for any baths in theories of quantum dissipation, namely, translational symmetry, approach to eventual thermal equilibrium, and complete positivity for the reduced density matrix of the system. The Agarwal bath has nice features of translational invariance and thermal equilibration. Although positivity is not guaranteed for certain initial conditions, it is for the large part not a problem. We first attach the Agarwal bath to our system of twin oscillators. Solutions to the dissipative dynamics of the Gaussian wavepackets are then utilized for separability studies.

Another trick in the bag of quantum dissipative theories is the phenomenology of the semigroup formalism, which is positivity-preserving by design and intrinsically quantum mechanical [15,16]. As a direct generalization of pure-dephasing for the one-oscillator model [1], we employ a form of Lindblad-type dissipation which preserves the system energy. Equations of motion are derived for first- and second-order moments from which the translational invariance is found to be broken, and the Schwinger boson representation is introduced to analyze the dynamics.

Most recently theories of quantum dissipation have been applied to quantum computations and quantum information manipulations. Although a majority of the work in the field is concerned with Hilbert spaces with finite dimensions, the interest has recently been expanded to continuous variable systems, in part because of the experimental realization of quantum teleportation of coherent states [4]. The concept of density matrix inseparability is of utmost importance. A separable density matrix can be represented by a mixture of direct-product states. Measurements made on separable states exhibit classical correlations. A bipartite density matrix is separable if it can be written in the form

$$\rho = \sum_i p_i \rho_A^i \otimes \rho_{\mathrm{B}}^i \,, \tag{1.1}$$

where $p_i > 0$, and $\sum_i p_i = 1$. An inseparability criterion concerning partial transpose of the density matrix was proposed by Peres [5], and later shown by Horodecki to be a necessary and sufficient condition for inseparability of 2×2 or 2×3 systems [6].

A major issue in quantum computer design is to overcome decoherence of the system of interest due to interactions with its environment. In this paper we construct decoherence free subspaces (DFSs) within a system of two oscillators submerged in a Lindblad-type bath. Although the subspaces as a whole are coupled to the bath by phase damping, dynamics with the subspaces are decoupled from the bath, thus protected from environment-induced dephasing which limits the storage time of quantum information.

The paper is organized as follows. In Section 2, we give a pedagogical introduction to quantum master equations and the semigroup formalism to quantum dissipation. In Section 3, we study the dissipative dynamics of two couple oscillators in an Agarwal bath. In Section 4, the Agarwal bath is replaced by a Lindblad-type bath

which preserves the total number of bosons in the system. Discussions are presented in Section 5.

2. Model and methodology

We are interested in the Brownian motion of quantum harmonic oscillators in a dissipative bath. For simplicity, we start with a model Hamiltonian describing only one primary oscillator of frequency ω_0 and mass m coupled to a bath of secondary oscillators of frequency ω_k and mass m_k $(k \neq 0)$

$$\hat{H}_A = \hbar\omega_0 a^\dagger a + \sum_k \hbar\omega_k b_k^\dagger b_k + q^s \sum_k g_k q_k^b \,, \tag{2.1}$$

where q^s and q_k^b are the coordinate observables for the system and the bath oscillators, respectively, which are related to the corresponding boson operators by

$$q^s = \left(\frac{\hbar}{2m\omega_0}\right)^{1/2}(a^\dagger + a), \qquad q_k^b = \left(\frac{\hbar}{2m_k\omega_k}\right)^{1/2}(b_k^\dagger + b_k) \tag{2.2}$$

and g_k are the coupling coefficients. The bath oscillators can be various phonon modes in a solid, or modes of vacuum radiation fields into which an excited atom decays via spontaneous emission. Adopting the rotating-wave approximation (RWA) widely used in fields such as quantum optics, the model Hamiltonian reduces to

$$\hat{H}_{\mathrm{RWA}} = \hbar\omega_0 a^\dagger a + \hbar\sum_k \omega_k b_k^\dagger b_k + \sum_k g_k(b_k^\dagger a + b_k a^\dagger) \,. \tag{2.3}$$

We note that the RWA neglects the rapidly oscillating terms of Eq. (2.1). For simplicity we shall set $\hbar = 1$ in the rest of the paper.

Agarwal has obtained a Schrödinger-representation master equation for the reduced density operator ρ in the limit of an infinite number of bath oscillators ($\sum_k \to \int \mathrm{d}\omega_k f(\omega_k)$) [14]

$$\frac{\partial\rho}{\partial t} + \mathrm{i}\omega[a^\dagger a, \rho] = \mathscr{L}_A(a, a^\dagger)\rho \equiv -\gamma\bar{n}[a + a^\dagger, [a + a^\dagger, \rho]]$$
$$-\gamma(a[a + a^\dagger, \rho] - [a + a^\dagger, \rho]a^\dagger - 2\rho) \,, \tag{2.4}$$

where $\gamma = \pi f(\omega_0)|g_c(\omega_0)|^2$ is the damping constant, $f(\omega)$ is the density of bath oscillators, $g_c(\omega)$ is the continuum form of g_k, $\bar{n} = (\mathrm{e}^{\hbar\omega/k_\mathrm{B}T} - 1)^{-1}$, ω is the renormalized frequency of ω_0 [17],

$$\omega = \omega_0 + \mathscr{P}\mathscr{P}\int_0^\infty \mathrm{d}\omega\, \frac{f(\omega)|g_c(\omega)|^2}{\omega - \omega_0} \tag{2.5}$$

and $\mathscr{P}\mathscr{P}$ stands for the Cauchy principal part. Approximations assumed in deriving Eq. (2.4) includes the Born approximation which treats the bath effects in the lowest order and the short memory hypothesis for the bath.

The purpose of this paper is to study of the effects of dissipation on the quantum dynamics of two coupled oscillators, and it is straightforward to generalize the

one-oscillator Agarwal master equation (2.4) to a system of two oscillators. If there are two primary oscillators coupled to each other in the system with a Hamiltonian

$$\hat{H}_{\text{two}} = \omega(a_1^\dagger a_1 + a_2^\dagger a_2) + J(a_1^\dagger a_2 + a_2^\dagger a_1)\,, \tag{2.6}$$

where J is the coupling strength, and each oscillator is dissipated by the bath modes in the form of Eq. (2.4), the master equation for the reduced density matrix for the two oscillators can be written as

$$\frac{\partial \rho}{\partial t} + \mathrm{i}[\hat{H}_{\text{two}}, \rho] = \sum_{i=1,2} \mathscr{L}_A(a_i, a_i^\dagger)\rho\,. \tag{2.7}$$

The Agarwal master equation (2.4) has found applications in a variety of fields despite of a well-known fact that it violates the positivity requirements of the reduced density matrix, and therefore leads to unphysical steady states for certain initial conditions [18–20]. To remedy the problem of positivity violation in the Agarwal master equation, Lindblad [21] showed that a completely positive map can be generated by

$$\mathscr{L}_L\rho = \sum_m [V_m, \rho V_m^\dagger] + [V_m\rho, V_m^\dagger] \tag{2.8}$$

from which the equation of motion of the density matrix can be expressed as

$$\frac{\partial \rho}{\mathrm{d}t} + \mathrm{i}[H, \rho] = \mathscr{L}_L\rho\,. \tag{2.9}$$

Here the V_m's are the Lindblad dissipation operators. For example, choosing a single dissipation operator [22]

$$V = \mu q + \mathrm{i}v p, \qquad V^\dagger = \mu q - \mathrm{i}v p\,, \tag{2.10}$$

where the operators q and p are defined as

$$\sqrt{2}q = a + a^\dagger\,, \tag{2.11}$$

$$\mathrm{i}\sqrt{2}p = a - a^\dagger\,, \tag{2.12}$$

one obtains

$$\frac{\partial \rho}{\mathrm{d}t} = -\mathrm{i}[H - 2\mu v pq, \rho] - \mu^2[q,[q,\rho]] - 2\mathrm{i}\mu v[q,[p,\rho]_+] - v^2[p,[p,\rho]]\,. \tag{2.13}$$

If two sets of dissipation operators are chosen instead [23]:

$$V_i = a_i q + b_i p, \qquad i = 1,2 \tag{2.14}$$

the master equation reads [23,24]

$$\begin{aligned}\frac{\partial \rho}{\partial t} = {} & -\mathrm{i}[\omega a^\dagger a + \varepsilon[q,p]_+, \rho] - \delta_1[q,[q,\rho]] - \delta_2[p,[p,\rho]] \\ & - \delta_0([q,[p,\rho]] + [p,[q,\rho]]) - \mathrm{i}\eta([q,[p,\rho]_+] - [p,[q,\rho]_+])\,.\end{aligned} \tag{2.15}$$

Variants of Eq. (2.15) can also be derived from generalized Weyl and Wigner transformations [24]. Eq. (2.15) can be used to describe, for instance, an electromagnetic field

mode interacting with an equilibrium bath of bosons in quantum optics, or dynamics of open systems in heavy ion collisions [23]. Due to the Schwartz inequality and the way Eq. (2.15) is constructed from the dissipation operators V_1 and V_2, Eq. (2.15) is a master equation of the Lindblad form if the following conditions are satisfied:

$$\delta_1, \delta_2 > 0 \,, \tag{2.16}$$

$$\delta_1 \delta_2 - \delta_0^2 \geqslant \eta/4 \,. \tag{2.17}$$

Compact expressions of exact solutions for (2.15) for all initial conditions have been obtained in Ref. [1].

A Lindblad-type bath was constructed very recently for a two-oscillator system [25] for separability studies. The Lindblad dissipation operators are chosen to be proportional to p_i and q_i of the two oscillators ($\sqrt{2}q_i = a_i + a_i^\dagger$, $\mathrm{i}\sqrt{2}p_i = a_i - a_i^\dagger$, $i = 1,2$). The Lindblad dissipation operators, however, can take forms other than linear in q and p [as in Eq. (2.14)]. For example, the equation of motion (2.4) has been generalized [26] to include an additional β term

$$\begin{aligned}\frac{\partial \rho}{\partial t} = &-\mathrm{i}\omega[a^\dagger a, \rho] - \beta[a^\dagger a, [a^\dagger a, \rho]] - \gamma\bar{n}[a + a^\dagger, [a + a^\dagger, \rho]] \\ &- \gamma(a[a + a^\dagger, \rho] - [a + a^\dagger, \rho]a^\dagger - 2\rho) \,,\end{aligned} \tag{2.18}$$

where the β-term can be derived from the semigroup formalism and by taking the system Hamiltonian ($a^\dagger a$) as the Lindblad dissipation operator. Similar applications of the semigroup formalism can be found in quantum optics (phase-damped oscillators) corresponding to a nondestructive measurement of photon number [27–29]. In this paper we also study the effect of Lindblad-type baths on the system of two oscillators. For example, one can use the Hermitian operators $a_1^\dagger a_1 + a_2^\dagger a_2$, and $a_1^\dagger a_2 + a_2^\dagger a_1$ as the Lindblad dissipation operators:

$$L_S\rho = -\beta\left[\sum_{i=1,2} a_i^\dagger a_i, \left[\sum_{i=1,2} a_i^\dagger a_i, \rho\right]\right] - \phi[a_1^\dagger a_2 + a_2^\dagger a_1, [a_1^\dagger a_2 + a_2^\dagger a_1, \rho]] \,. \tag{2.19}$$

The β-term is a direct generalization of (2.18) while the ϕ-term is the off-diagonal coupling describing the dephasing of bosons as they hop.

Semigroup theories also showed that for harmonic oscillators density-matrix positivity, translational invariance and approach to thermal equilibrium cannot be satisfied simultaneously. The semigroup formalism in general does not guarantee approach of thermal equilibrium without extra constraints. Detailed balance, however, can be imposed onto Eq. (2.15) in forms of parameter constraints to ensure final approach to equilibrium in the framework of semigroup theories. Efforts along this direction have been a matter of much recent interest [22,28,30]. For instance, Gao has proposed a master equation constructed from one single Lindblad dissipation operator which is linear in both a and $a^\dagger$ with he proper proportionality coefficients to sustain detailed balance [22]. The positivity requirement in Gao's approach is marginally satisfied with $\delta_1\delta_2 = \eta^2/4$. Gao's construction of L_D via a single Lindblad dissipation

operator has recently found support by a first-principle derivation of master equations for collision-driven dissipative evolution [30].

In quantum optics, atomic physics and chemical physics, master equations often need to be solved by numerical integration which may involve continuous evolution of the density matrix as well as jumps at random instances [31]. Here we adopt a method of solution which utilizes the quantum characteristic function $\chi(\lambda_1,\lambda_1^*,\lambda_2,\lambda_2^*,t)$ [32]

$$\chi(\lambda_1,\lambda_1^*,\lambda_2,\lambda_2^*,t)=\mathrm{Tr}(\rho e^{\lambda_1 a_1^\dagger}e^{-\lambda_1^* a_1}e^{\lambda_2 a_2^\dagger}e^{-\lambda_2^* a_2}), \tag{2.20}$$

where the trace is taken over the system of two oscillators. Instead of dealing directly with the density matrix, we derive an equation of motion for the characteristic function $\chi(\lambda_1,\lambda_1^*,\lambda_2,\lambda_2^*,t)$ which is then solved by method of characteristics.

The two-oscillator characteristic function $\chi(\lambda_1,\lambda_1^*,\lambda_2,\lambda_2^*)$ is a direct generalization of the characteristic function for one oscillator $\chi^1(\lambda,\lambda^*)$ [33]:

$$\chi^1(\lambda,\lambda^*)=\mathrm{Tr}(\rho e^{\lambda a^\dagger}e^{-\lambda^* a}). \tag{2.21}$$

For one oscillator, the quantum characteristic function $\chi^1(\lambda,\lambda^*)$ is the Fourier transform of the phase space distribution function $P_1(z,z^*)$ of the density matrix ρ

$$\chi^1(\lambda,\lambda^*)=\int d^2z\exp(\lambda z^*-\lambda^* z)P_1(z,z^*). \tag{2.22}$$

The phase space distribution function $P_1(z,z^*)$ is also called Glauber–Sudarshan P representation of the density matrix ρ which plays the role of a quasiprobability:

$$\rho=\int d^2z P_1(z,z^*)|z\rangle\langle z|. \tag{2.23}$$

For the case of two oscillators similar relations exist between the characteristic function $\chi(\lambda_1,\lambda_1^*,\lambda_2,\lambda_2^*)$ and the phase space distribution function $P(z_1,z_1^*,z_2,z_2^*)$ of the density matrix ρ:

$$\begin{aligned}&\chi(\lambda_1,\lambda_1^*,\lambda_2,\lambda_2^*)\\&\quad=\int d^2z_1\,d^2z_2\exp(\lambda_1 z_1^*-\lambda_1^* z_1+\lambda_2 z_2^*-\lambda_2^* z_2)P(z_1,z_1^*,z_2,z_2^*).\end{aligned} \tag{2.24}$$

The often-used Wigner distribution function $W(z_1,z_1^*,z_2,z_2^*)$ is the Fourier transform of the Wigner characteristic function $\chi^{\mathrm{W}}(\lambda_1,\lambda_1^*,\lambda_2,\lambda_2^*)$ which is related to $\chi(\lambda_1,\lambda_1^*,\lambda_2,\lambda_2^*)$ by a simple factor [34]:

$$\begin{aligned}&\chi^{\mathrm{W}}(\lambda_1,\lambda_1^*,\lambda_2,\lambda_2^*)\\&\quad=\int d^2z_1\,d^2z_2\exp(\lambda_1 z_1^*-\lambda_1^* z_1+\lambda_2 z_2^*-\lambda_2^* z_2)W(z_1,z_1^*,z_2,z_2^*)\end{aligned} \tag{2.25}$$

$$=e^{-1/2|\lambda_1|^2-1/2|\lambda_2|^2}\chi(\lambda_1,\lambda_1^*,\lambda_2,\lambda_2^*). \tag{2.26}$$

The Wigner distribution function $W(z_1,z_1^*,z_2,z_2^*)$ gives a description of quantum states in close resemblance to the phase-space classical descriptions, which is related to the Weyl classical-quantum correspondence [35].

We wrap up this section with an illustration of our method of solutions for the problem of one oscillator in an Agarwal bath [cf. Eq. (2.4)]. An equation for $\chi^1(\lambda,\lambda^*)$ follows from the quantum master equation (2.4):

$$\frac{\partial\chi^1}{\partial t}+[-\mathrm{i}\omega\lambda+\gamma(\lambda+\lambda^*)]\frac{\partial\chi^1}{\partial\lambda}+[\mathrm{i}\omega\lambda^*+\gamma(\lambda+\lambda^*)]\frac{\partial\chi^1}{\partial\lambda^*}$$
$$=-\gamma\bar{n}(\lambda+\lambda^*)^2\chi^1\ . \tag{2.27}$$

Eq. (2.27) is solved by the method of characteristics [36]. Assuming the characteristic function

$$\chi^1(\lambda,\lambda^*)=\exp\left[\sum_{mn}C_{mn}(t)\lambda^m(-\lambda^*)^n\right]\ , \tag{2.28}$$

where $C_{mn}(t)$ are the coefficients to be determined, one arrives at the set of differential equations for C_{mn}:

$$\dot{C}_{10}=(\mathrm{i}\omega-\gamma)C_{10}+\gamma C_{01}\ , \tag{2.29}$$

$$\dot{C}_{01}=(-\mathrm{i}\omega-\gamma)C_{01}+\gamma C_{10}\ , \tag{2.30}$$

$$\dot{C}_{20}=2(\mathrm{i}\omega-\gamma)C_{20}-\gamma(\bar{n}-C_{11})\ , \tag{2.31}$$

$$\dot{C}_{11}=2\gamma(\bar{n}-C_{11})+2\gamma(C_{02}+C_{20})\ , \tag{2.32}$$

$$\dot{C}_{02}=-2(\mathrm{i}\omega+\gamma)C_{02}-\gamma(\bar{n}-C_{11})\ . \tag{2.33}$$

Analytical solutions for all initial conditions are conveniently listed in Appendix A.

3. Two oscillators in an Agarwal bath

In this section, we apply the quantum characteristic functions introduced in the previous section to a system of two coupled harmonic oscillators in an Agarwal bath. Our focus is to derive a set of general equations for the time evolution of the Gaussian wavepackets with arbitrary initial means and variances.

From the master equation for the reduced density matrix for the two oscillators one can derive an equation of motion for the characteristic function $\chi(\lambda_1,\lambda_1^*,\lambda_2,\lambda_2^*,t)$:

$$\frac{\partial\chi}{\partial t}+\sum_{i=1,2}\left[\left(-\mathrm{i}\omega\lambda_i\frac{\partial\chi}{\partial\lambda_i}+\mathrm{i}\omega\lambda_i^*\frac{\partial\chi}{\partial\lambda_i^*}\right)\right.$$
$$\left.+\gamma(\lambda_i+\lambda_i^*)\left(\frac{\partial\chi}{\partial\lambda_i}+\frac{\partial\chi}{\partial\lambda_i^*}\right)+\gamma\bar{n}(\lambda_i+\lambda_i^*)^2\chi\right]$$
$$-\mathrm{i}J\left(\lambda_2\frac{\partial}{\partial\lambda_1}+\lambda_1\frac{\partial}{\partial\lambda_2}\right)\chi+\mathrm{i}J\left(\lambda_2^*\frac{\partial}{\partial\lambda_1^*}+\lambda_1^*\frac{\partial}{\partial\lambda_2^*}\right)\chi=0\ . \tag{3.1}$$

We assume that the characteristic function has the form

$$\chi(\lambda_1,\lambda_1^*,\lambda_2,\lambda_2^*,t)=\exp\left[\sum_{mnkl}C_{mn,kl}\lambda_1^m(-\lambda_1^*)^n\lambda_2^k(-\lambda_2^*)^l\right]\ . \tag{3.2}$$

The Gaussian wavepackets are obtained by restricting $m+n+k+l\leqslant 2$ in the summation over m,n,k,l. Aside from the first- and second-order moments introduced for each individual oscillator, four more cross moments, namely, $C_{10,01}$, $C_{01,10}$, $C_{10,10}$, and $C_{01,01}$, are added to account for cross correlations of the two oscillators. The second-order terms in the exponent of Eq. (3.2) can be put in a matrix form:

$$\frac{1}{2}(\lambda_1^*\ \lambda_1\ \lambda_2^*\ \lambda_2)\begin{pmatrix}-C_{11,00} & 2C_{02,00} & -C_{10,01} & C_{10,10}\\ 2C_{20,00} & -C_{11,00} & C_{10,10} & -C_{10,01}\\ -C_{10,01} & C_{01,01} & -C_{00,11} & 2C_{00,02}\\ C_{10,10} & -C_{01,10} & 2C_{00,20} & -C_{00,11}\end{pmatrix}\begin{pmatrix}\lambda_1\\ \lambda_1^*\\ \lambda_2\\ \lambda_2^*\end{pmatrix}\ . \tag{3.3}$$

One can also write the second-order exponent in the form

$$\frac{1}{2}(\lambda_1^R\ \lambda_1^I\ \lambda_2^R\ \lambda_2^I)M(\lambda_1^R\ \lambda_1^I\ \lambda_2^R\ \lambda_2^I)^{\mathrm{T}}\ , \tag{3.4}$$

where the new basis is related to $(\lambda_1^*\ \lambda_1\ \lambda_2^*\ \lambda_2)^{\mathrm{T}}$ by

$$\begin{pmatrix}\lambda_1^R\\ \lambda_1^I\\ \lambda_2^R\\ \lambda_2^I\end{pmatrix}=\begin{pmatrix}\frac{1}{2} & \frac{1}{2} & 0 & 0\\ -\frac{i}{2} & \frac{i}{2} & 0 & 0\\ 0 & 0 & \frac{1}{2} & \frac{1}{2}\\ 0 & 0 & -\frac{i}{2} & \frac{i}{2}\end{pmatrix}\begin{pmatrix}\lambda_1\\ \lambda_1^*\\ \lambda_2\\ \lambda_2^*\end{pmatrix} \tag{3.5}$$

and M is a real symmetric matrix

$$M=\begin{pmatrix}V_1 & V_{12}\\ V_{12}^T & V_2\end{pmatrix} \tag{3.6}$$

with

$$V_1=2\begin{pmatrix}C_{20,00}+C_{02,00}-C_{11,00} & \mathrm{i}(C_{20,00}-C_{02,00})\\ \mathrm{i}(C_{20,00}-C_{02,00}) & -C_{02,00}-C_{20,00}-C_{11,00}\end{pmatrix}\ , \tag{3.7}$$

$$V_2=2\begin{pmatrix}C_{00,20}+C_{00,02}-C_{00,11} & \mathrm{i}(C_{00,20}-C_{00,02})\\ \mathrm{i}(C_{00,20}-C_{00,02}) & -C_{00,02}-C_{00,20}-C_{00,11}\end{pmatrix} \tag{3.8}$$

and

$$V_{12}=\begin{pmatrix}C_{10,10}+C_{01,01}-C_{10,01}-C_{01,10} & \mathrm{i}(C_{10,10}-C_{01,01}+C_{10,01}-C_{01,10})\\ \mathrm{i}(C_{10,10}-C_{01,01}-C_{10,01}+C_{01,10}) & -C_{10,10}-C_{01,01}-C_{10,01}-C_{01,10}\end{pmatrix}\ . \tag{3.9}$$

The corresponding Wigner characteristic function $\chi^{\mathrm{W}}(\lambda_1, \lambda_1^*, \lambda_2, \lambda_2^*)$ can also be written in a matrix form

$$\chi^{\mathrm{W}}(\lambda_1^R, \lambda_1^I, \lambda_2^R, \lambda_2^I) = \exp\left[-\frac{1}{2}(\lambda_1^R \ \lambda_1^I \ \lambda_2^R \ \lambda_2^I) W (\lambda_1^R \ \lambda_1^I \ \lambda_2^R \ \lambda_2^I)^{\mathrm{T}}\right], \tag{3.10}$$

where the matrix W

$$W = I - M \tag{3.11}$$

with I the identity matrix.

Various operator averages can be calculated from the first and second moments, for example, for the first oscillator, one has

$$\langle a_1 \rangle = C_{01,00}\,, \tag{3.12}$$

$$\langle a_1^\dagger \rangle = C_{10,00}\,, \tag{3.13}$$

$$\langle a_1^2 \rangle = 2C_{02,00} + (C_{01,00})^2\,, \tag{3.14}$$

$$\langle a_1^{\dagger 2} \rangle = 2C_{20,00} + (C_{10,00})^2\,, \tag{3.15}$$

$$\langle a_1^\dagger a_1 \rangle = C_{11,00} + C_{10,00}C_{01,00} \tag{3.16}$$

similar relations exist for the second oscillator; and for cross correlations:

$$\langle a_1^\dagger a_2 \rangle = C_{10,00}C_{00,01} + C_{10,01}\,, \tag{3.17}$$

$$\langle a_2^\dagger a_1 \rangle = C_{01,00}C_{00,10} + C_{01,10}\,, \tag{3.18}$$

$$\langle a_1^\dagger a_2^\dagger \rangle = C_{10,00}C_{00,10} + C_{10,10}\,, \tag{3.19}$$

$$\langle a_1 a_2 \rangle = C_{01,00}C_{00,01} + C_{01,01}\,. \tag{3.20}$$

Equations of motion for the moments can be derived from (3.1). The first-order moments obey four coupled differential equations:

$$\dot{C}_{10,00} = (\mathrm{i}\omega - \gamma)C_{10,00} + \gamma C_{01,00} + \mathrm{i}JC_{00,10}\,, \tag{3.21}$$

$$\dot{C}_{01,00} = (-\mathrm{i}\omega - \gamma)C_{01,00} + \gamma C_{10,00} - \mathrm{i}JC_{00,01}\,, \tag{3.22}$$

$$\dot{C}_{00,10} = (\mathrm{i}\omega - \gamma)C_{00,10} + \gamma C_{00,01} + \mathrm{i}JC_{10,00}\,, \tag{3.23}$$

$$\dot{C}_{00,01} = (-\mathrm{i}\omega - \gamma)C_{00,01} + \gamma C_{00,10} - \mathrm{i}JC_{01,00}\,. \tag{3.24}$$

Eqs. (3.21)–(3.24) obey the Ehrenfest theorem which expresses a formal connection between the time dependence of expectation values of canonically conjugate variables and the Hamiltonian equations of classical mechanics:

$$\frac{\mathrm{d}}{\mathrm{d}t}(C_{10,00} + C_{01,00} + C_{00,10} + C_{00,01}) = \mathrm{i}(\omega + J)(C_{10,00} - C_{01,00} + C_{00,10} - C_{00,01})\,. \tag{3.25}$$

Second-order moments including the cross moments follow:

$$\dot{C}_{20,00} = 2(\mathrm{i}\omega - \gamma)C_{20,00} - \gamma(\bar{n} - C_{11,00}) + \mathrm{i}JC_{10,10} \,, \tag{3.26}$$

$$\dot{C}_{11,00} = 2\gamma(\bar{n} - C_{11,00}) + 2\gamma(C_{02,00} + C_{20,00}) + \mathrm{i}J(C_{01,10} - C_{10,01}) \,, \tag{3.27}$$

$$\dot{C}_{02,00} = -2(\mathrm{i}\omega + \gamma)C_{02,00} - \gamma(\bar{n} - C_{11,00}) - \mathrm{i}JC_{01,01} \,, \tag{3.28}$$

$$\dot{C}_{10,10} = 2\mathrm{i}\omega C_{10,10} + 2\mathrm{i}J(C_{20,00} + C_{00,20}) + \gamma(C_{10,01} + C_{01,10} - 2C_{10,10}) \,, \tag{3.29}$$

$$\dot{C}_{10,01} = \mathrm{i}J(C_{00,11} - C_{11,00}) + \gamma(C_{01,01} + C_{10,10} - 2C_{10,01}) \,, \tag{3.30}$$

$$\dot{C}_{01,10} = -\mathrm{i}J(C_{00,11} - C_{11,00}) + \gamma(C_{01,01} + C_{10,10} - 2C_{01,10}) \,, \tag{3.31}$$

$$\dot{C}_{01,01} = -2\mathrm{i}\omega C_{01,01} - 2\mathrm{i}J(C_{02,00} + C_{00,02}) + \gamma(C_{10,01} + C_{01,10} - 2C_{01,01}) \,, \tag{3.32}$$

$$\dot{C}_{00,20} = 2(\mathrm{i}\omega - \gamma)C_{00,20} - \gamma(\bar{n} - C_{00,11}) + \mathrm{i}JC_{10,10} \,, \tag{3.33}$$

$$\dot{C}_{00,11} = 2\gamma(\bar{n} - C_{00,11}) + 2\gamma(C_{00,02} + C_{00,20}) - \mathrm{i}J(C_{01,10} - C_{10,01}) \,, \tag{3.34}$$

$$\dot{C}_{00,02} = -2(\mathrm{i}\omega + \gamma)C_{00,02} - \gamma(\bar{n} - C_{00,11}) - \mathrm{i}JC_{01,01} \,. \tag{3.35}$$

For a state with initial zero first-order moments, the dissipative dynamics and the properties of the Gaussian density matrix as a function of time are completely determined by the above ten coupled equations.

An inseparability criterion based on the combined variance of a pair of Einstein–Podolsky–Rosen (EPR) type operators was proposed by Duan et al. [8] for continuous variable systems. A maximally entangled state of continuous variables can be written as a co-eigenstate of a pair of EPR type operators resulting in a zero combined variance of the two operators [37]. However, for any separable state, a lower bound exists to the combined variance of the two EPR type operators. The separability criterion of Duan et al. can be expressed conveniently in terms of the second-order moments as follows. Consider two EPR type operators:

$$\hat{u} = rq_1 + r^{-1}q_2 \,, \tag{3.36}$$

$$\hat{v} = rp_1 - r^{-1}p_2 \,, \tag{3.37}$$

where r is a nonzero real number, and the operators q_i and p_i $(i = 1, 2)$ are defined as

$$\sqrt{2}q_i = a_i + a_i^{\dagger} \,, \tag{3.38}$$

$$\mathrm{i}\sqrt{2}p_i = a_i - a_i^{\dagger} \,. \tag{3.39}$$

The sufficient criterion for inseparability is that the variances of the operators $\hat{u}$ and $\hat{v}$ satisfy

$$\langle(\Delta\hat{u})^2\rangle + \langle(\Delta\hat{v})^2\rangle \leqslant r^2 + r^{-2} \,. \tag{3.40}$$

For Gaussian states (as the case of two harmonic oscillators in an Agarwal bath investigated in this section) the condition (3.40) is also necessary only if the parameter

r is chosen in a particular way [8]. For general r, condition (3.40) is not necessary. The variances of the operators $\hat{u}$ and $\hat{v}$ can be written in the second-order moments:

$$\begin{aligned}\langle(\Delta\hat{u})^2\rangle = \frac{1}{2}(r^2 + r^{-2}) &+ r^2(C_{02,00} + C_{20,00} + C_{11,00}) \\ &+ r^{-2}(C_{00,02} + C_{00,20} + C_{00,11}) \\ &+ C_{01,01} + C_{10,10} + C_{10,01} + C_{01,10}\,, \end{aligned} \tag{3.41}$$

$$\begin{aligned}\langle(\Delta\hat{v})^2\rangle = \frac{1}{2}(r^2 + r^{-2}) &- r^2(C_{02,00} + C_{20,00} - C_{11,00}) \\ &- r^{-2}(C_{00,02} + C_{00,20} - C_{00,11}) \\ &+ C_{01,01} + C_{10,10} - C_{10,01} - C_{01,10}\,. \end{aligned} \tag{3.42}$$

Therefore their sum is

$$\begin{aligned}&\langle(\Delta\hat{u})^2\rangle + \langle(\Delta\hat{v})^2\rangle \\ &\quad = r^2 + r^{-2} + 2r^2C_{11,00} + 2r^{-2}C_{00,11} + 2(C_{01,01} + C_{10,10})\,. \end{aligned} \tag{3.43}$$

The sufficient condition for inseparability is then

$$\begin{aligned}&\frac{1}{2}[\langle(\Delta\hat{u})^2\rangle + \langle(\Delta\hat{v})^2\rangle - (r^2 + r^{-2})] \\ &\quad = r^2C_{11,00} + r^{-2}C_{00,11} + C_{01,01} + C_{10,10} \leqslant 0\,. \end{aligned} \tag{3.44}$$

For $r = 1$, this requires

$$\begin{aligned}S &\equiv \frac{1}{2}[\langle(\Delta\hat{u})^2\rangle + \langle(\Delta\hat{v})^2\rangle - (r^2 + r^{-2})]|_{r=1} \\ &= C_{10,10} + C_{01,01} + C_{11,00} + C_{00,11} \leqslant 0\,. \end{aligned} \tag{3.45}$$

This means that the system is inseparable whenever the sum of the four second-order moments, $C_{10,10} + C_{01,01} + C_{11,00} + C_{00,11}$, is not positive. It is also interesting to look at the difference between the two variances

$$\begin{aligned}&\langle(\Delta\hat{u})^2\rangle - \langle(\Delta\hat{v})^2\rangle \\ &\quad = 2r^2(C_{02,00} + C_{20,00}) + 2r^{-2}(C_{00,02} + C_{00,20}) + 2(C_{10,01} + C_{01,10})\,. \end{aligned} \tag{3.46}$$

If $r = 1$, then one half of this difference is

$$\begin{aligned}A &\equiv \frac{1}{2}[\langle(\Delta\hat{u})^2\rangle - \langle(\Delta\hat{v})^2\rangle]|_{r=1} \\ &= C_{20,00} + C_{02,00} + C_{00,02} + C_{00,20} + C_{10,01} + C_{01,10}\,. \end{aligned} \tag{3.47}$$

So it turns out that while S is the sum of four out of the ten second-order moments defined in Eq. (3.3), A is the sum of the remaining six second-order moments. From the set of coupled equations for second moments one can derive equations of motion

for S in Eq. (3.45) and A in Eq. (3.47):

$$\dot{S} = -2\gamma(S - 2\bar{n}) + 2\gamma A + 2\mathrm{i}J(C_{20,00} - C_{02,00} + C_{00,20} - C_{00,02}) + 2\mathrm{i}\omega(C_{10,10} - C_{01,01}) , \tag{3.48}$$

$$\dot{A} = -2\gamma A + 2\gamma(S - 2\bar{n}) + 2\mathrm{i}J(C_{10,10} - C_{01,01}) + 2\mathrm{i}\omega(C_{20,00} - C_{02,00} + C_{00,20} - C_{00,02}) . \tag{3.49}$$

It is obvious that S asymptotically goes to $2\bar{n}$ at long times, and A to zero. Adding the above two equations,

$$\dot{S} + \dot{A} = 2\mathrm{i}(J + \omega)(C_{20,00} - C_{02,00} + C_{00,20} - C_{00,02} + C_{10,10} - C_{01,01}) . \tag{3.50}$$

Now we are in a position to look at some examples. One of the most frequently mentioned example in quantum teleportation is the two-oscillator squeezed state which was also utilized in the experimental realization of continuous-variable teleportation [4]. If the initial state is the highly entangled two-oscillator squeezed state

$$\mathrm{e}^{-s(a_1^\dagger a_2^\dagger - a_1 a_2)}|0\rangle \tag{3.51}$$

with s a real number and $|0\rangle$ the vacuum state for both oscillators, the characteristic function at time $t = 0$ has the form

$$\chi(\lambda_1, \lambda_1^*, \lambda_2, \lambda_2^*, 0) = \exp\left[-\frac{1}{2}\sinh 2s(\lambda_1\lambda_2 + \lambda_1^*\lambda_2^*) - \sinh^2 s(|\lambda_1|^2 + |\lambda_2|^2)\right] . \tag{3.52}$$

Details of the derivation of Eq. (3.52) are given in Appendix B. Therefore, at $t = 0$ there are only four nonzero second moments, namely,

$$C_{11,00}(0) = C_{00,11}(0) = \sinh^2 s , \tag{3.53}$$

$$C_{10,10}(0) = C_{01,01}(0) = -\tfrac{1}{2}\sinh 2s . \tag{3.54}$$

The pure state $\mathrm{e}^{-s(a_1^\dagger a_2^\dagger - a_1 a_2)}|0\rangle$ at $t = 0$ has an S value

$$S = 2\sinh^2 s - \sinh 2s = \mathrm{e}^{-2s} - 1 . \tag{3.55}$$

The sufficient condition for inseparability (3.45) is satisfied at $t = 0$ for $s > 0$. This is not true, however, for $s < 0$. One needs to seek another pair of EPR type operators. We pick

$$\hat{u}' = rq_1 - r^{-1}q_2 , \tag{3.56}$$

$$\hat{v}' = rp_1 + r^{-1}p_2 . \tag{3.57}$$

For this pair of EPR type operators the sufficient condition for inseparability is then

$$\frac{1}{2}[\langle(\Delta\hat{u}')^2\rangle + \langle(\Delta\hat{v}')^2\rangle - (r^2 + r^{-2})] = r^2 C_{11,00} + r^{-2}C_{00,11} - C_{01,01} - C_{10,10} \leqslant 0 . \tag{3.58}$$

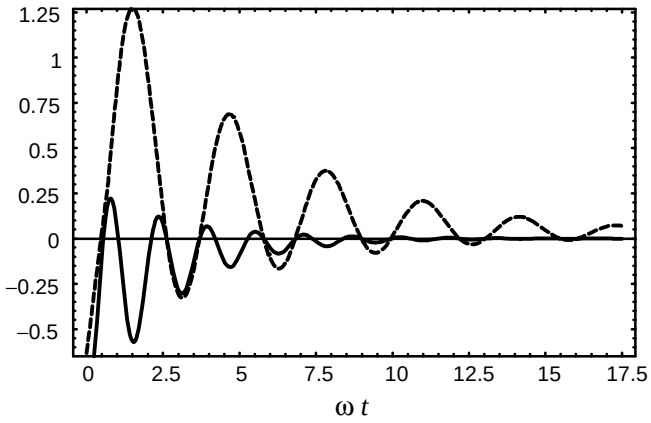

Fig. 1. The product of S and D (solid) and S (dashed) are plotted as a function of ωt for $s=\frac{1}{2}$, $\bar{n}=0.01$, $J=0$, and $\gamma/\omega=0.1$. The system remains inseparable until $\omega t=17.34$.

For $r=1$, this requires

$$D \equiv C_{11,00}+C_{00,11}-C_{10,10}-C_{01,01} \leqslant 0\,, \tag{3.59}$$

which shows the state with a negative s is also inseparable.

In Fig. 1, we plot both the product of S and D (solid line) and S (dashed line) for a case $s=\frac{1}{2}$, $\bar{n}=0.01$, $J=0$, and $\gamma/\omega=0.1$. Most of the time S and D are alternatively negative although there are some time intervals within which both are positive. S and D cannot be negative at the same time because that would falsely imply

$$\langle a_1^{\dagger}a_1\rangle+\langle a_2^{\dagger}a_2\rangle = C_{11,00}+C_{00,11}<0\,. \tag{3.60}$$

Since either one of the two being negative is sufficient to label the system as inseparable, the two-mode squeezed state is entangled for all the time interval plotted in Fig. 1 (as will be shown in Fig. 2 the system remains inseparable until $\omega t=17.34$).

The Peres–Horodecki condition for inseparability concerns the positivity of partial transpose (PPT) of the density matrix. When it is applied to Gaussian states in a two-oscillator system, it can be written in the form [10]

$$\Xi(t)<0\,, \tag{3.61}$$

where the auxiliary function $\Xi(t)$ is defined as

$$\begin{aligned}\Xi(t) \equiv{}& \mathrm{Det}[I-V_1]\,\mathrm{Det}[I-V_2]\\ &+(1-|\mathrm{Det}[V_{12}]|)^2-\mathrm{Tr}[(I-V_1)KV_{12}K(I-V_2)V_{12}^{T}K]\\ &-\mathrm{Det}[I-V_1]-\mathrm{Det}[I-V_2]\end{aligned} \tag{3.62}$$

with the matrix K

$$K=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\,. \tag{3.63}$$

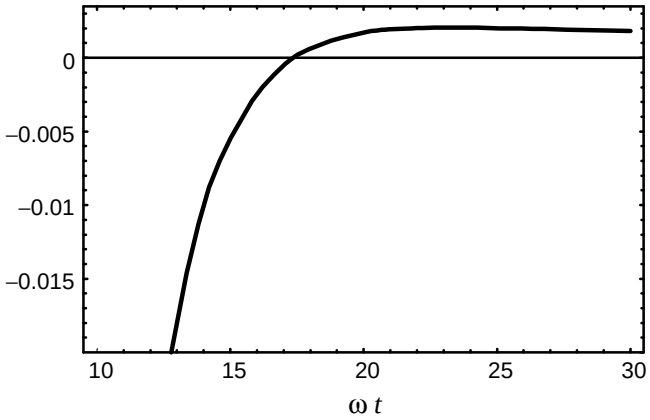

Fig. 2. $\Xi(t)$ as a function of ωt for $s=1/2$, $\bar{n}=0.01$, $J=0$, and $\gamma/\omega=0.1$. The system remains inseparable until $\omega t=17.34$.

The Peres–Horodecki condition is both necessary and sufficient for inseparability for the system of two harmonic oscillators in an Agarwal bath (for which the wavepacket is Gaussian) as shown by Simon [10]. Since the determinant of $I-M$ can be written as

$$\begin{aligned}\mathrm{Det}[I-M] &= \mathrm{Det}[I-V_1]\,\mathrm{Det}[I-V_2] \\ &\quad + \mathrm{Det}^2[V_{12}] - \mathrm{Tr}[(I-V_1)KV_{12}K(I-V_2)V_{12}^T K] \end{aligned} \tag{3.64}$$

the inseparability condition is equivalent to

$$\mathrm{Det}[M-I] + 2|\mathrm{Det}[V_{12}]| - \mathrm{Det}[V_1-I] - \mathrm{Det}[V_2-I] > 1\ . \tag{3.65}$$

In Fig. 2 we plot $\Xi(t)$ for the same case as in Fig. 1 with $s=1/2$, $\bar{n}=0.01$, $J=0$. The system is inseparable until ωt reaches about 17.34, which confirms our earlier conclusions drawn from Fig. 1 using the criterion of Duan et al. At this low temperature the system is inseparable for most of the time leading to thermal equilibrium.

In Fig. 3 we plot $\Xi(t)$ for a high temperature $\bar{n}=0.5$ with the rest of parameters unchanged from Fig. 2. Thermal fields render the system separable at earlier times as the temperature increases. The asymptotic values of $\Xi(t)$ at long times also increase rapidly with the bath temperatures. This can be shown by substituting the asymptotic form of the M matrix

$$M(\infty) = \mathrm{diag}(2\bar{n}+1, 2\bar{n}+1, 2\bar{n}+1, 2\bar{n}+1) \tag{3.66}$$

into Eq. (3.62) which leads to

$$\Xi(\infty) = 16\bar{n}^2(1+\bar{n})^2\ . \tag{3.67}$$

This means that for any finite temperatures it takes a finite amount of time to reach the separable thermalized state from an initial two-mode squeezed state, while for zero temperature the initial two-mode squeezed state will always stay inseparable until $t=\infty$.

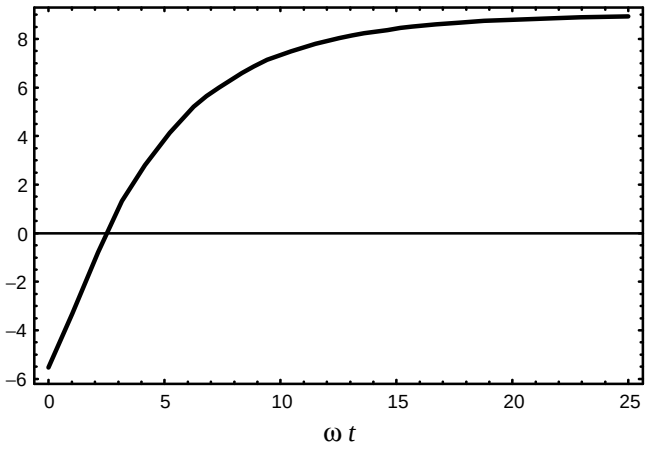

Fig. 3. $\Xi(t)$ as a function of ωt for $s = 1/2$, $\bar{n} = 0.5$, $J = 0$, and $\gamma/\omega = 0.1$. The system remains inseparable until $\omega t = 2.50$.

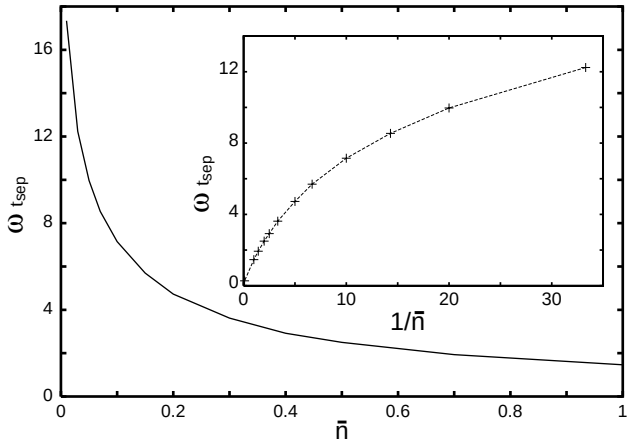

Fig. 4. In the main figure, ωt_{sep} is plotted as a function of $\bar{n}$ for $\gamma/\omega = 0.1$ and $s = 1/2$. In the inset, ωt_{sep} is plotted versus $1/\bar{n}$.

Starting from the two-mode squeezed state, the original inseparable state will become separable at time t_{sep} which depends sensitively on the bath temperature T. In Fig. 4 we display ωt_{sep} as a function of $\bar{n}$. At zero temperature $t_{\text{sep}} = \infty$, while t_{sep} approaches 0 as T tends to ∞. In the inset of Fig. 4 ωt_{sep} is plotted versus $1/\bar{n}$ shows t_{sep} at high temperatures asymptotically goes to zero.

So far the pair of oscillators is decoupled ($J = 0$). In Fig. 5 we plot $\Xi(t)$ after adding a finite value of the coupling strength ($J = 0.5$) to the decoupled oscillators. The upper panel corresponds to $\bar{n} = 0.01$, $s = 1/2$, and the lower panel, $\bar{n} = 0.5$, $s = 1/2$.

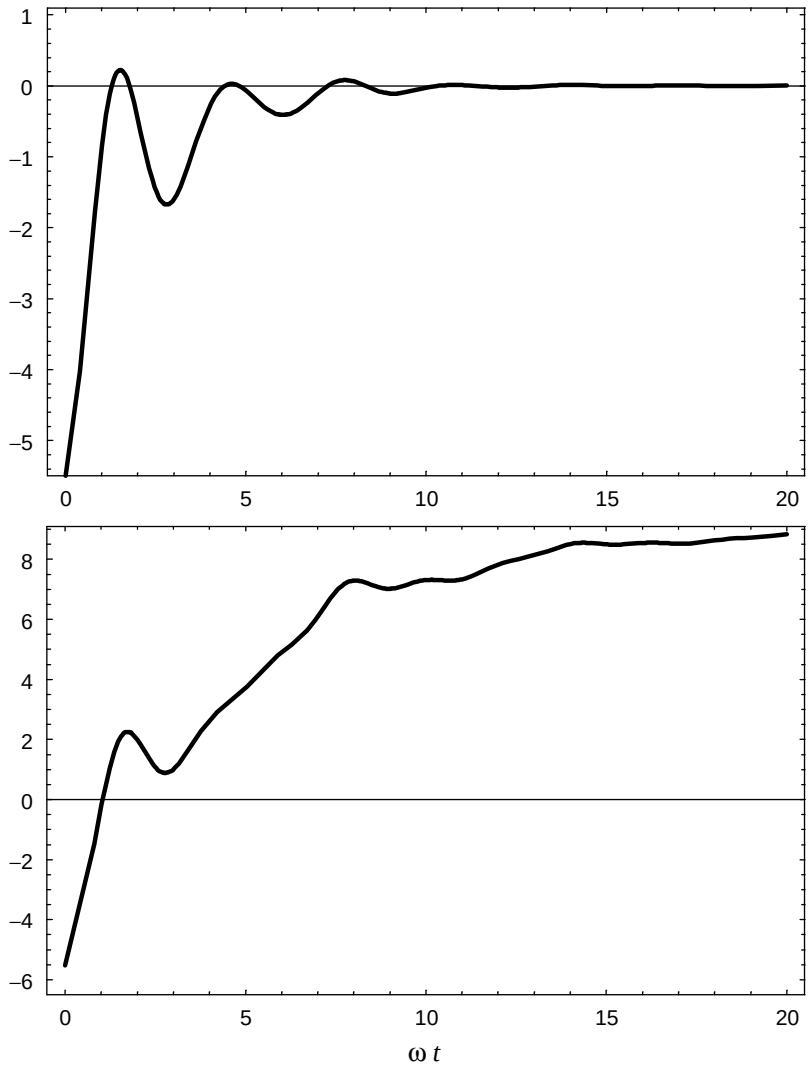

Fig. 5. $\Xi(t)$ as a function of ωt for $s = 1/2$, $J = 0.5$, and $\gamma/\omega = 0.1$. Two temperatures are plotted, $\bar{n} = 0.01$ (upper panel), and $\bar{n} = 0.5$ (lower panel).

It is interesting that for finite J and at low temperatures the system can go from being inseparable to being separable and again back to being inseparable. While for the $J = 0$ cases, $\Xi(t)$ is an even function of the squeezing parameter s, for nonzero J's, $\Xi(t)$ changes as s switches signs. In Fig. 6 we plot $\Xi(t)$ for $s = -1/2$ with the rest of parameters identical to those in Fig. 5. For both temperatures $\bar{n} = 0.01$ and 0.5, $\Xi(t)$ differs from that in Fig. 5.

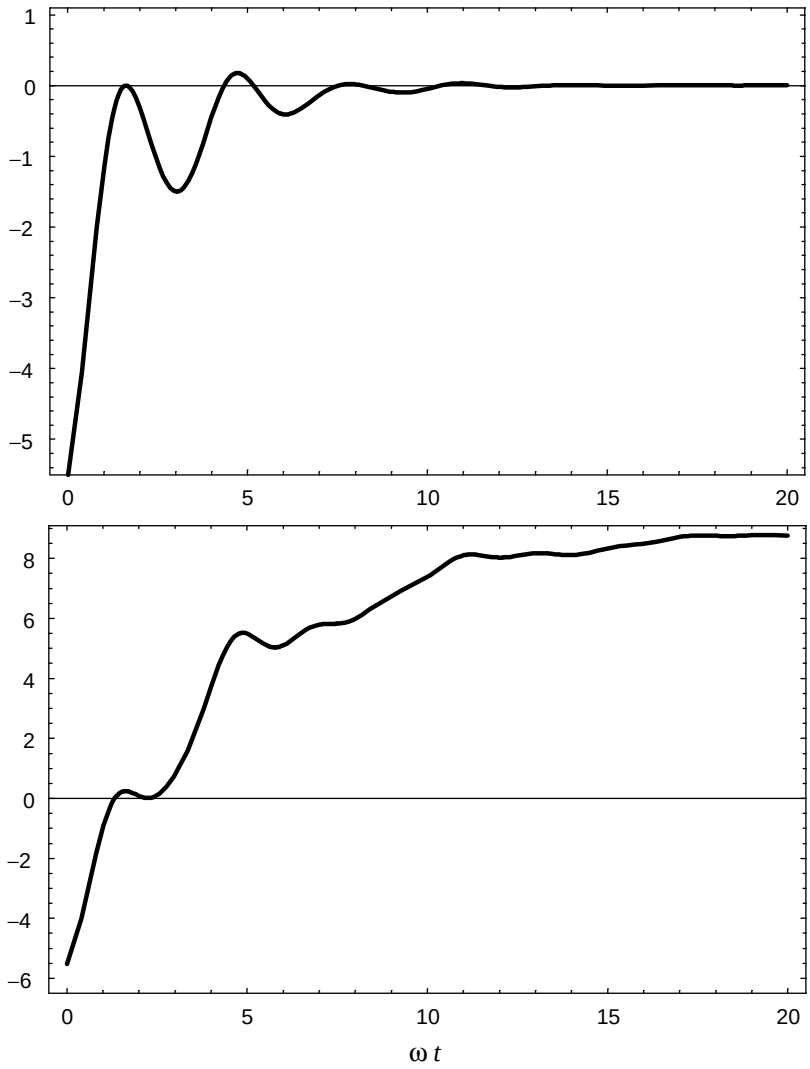

Fig. 6. $\Xi(t)$ as a function of ωt for $s=-1/2$, $J=0.5$, and $\gamma/\omega=0.1$. Two temperatures are plotted, $\bar{n}=0.01$ (upper panel), and $\bar{n}=0.5$ (lower panel).

4. Dissipation via semigroup

Apart from the on-site Agarwal-type dissipation discussed in Section 3, various Lindblad-type dissipation mechanisms can occur in the system of two coupled oscillators. For example, one can use the Hermitian operators $a_1^\dagger a_1 + a_2^\dagger a_2$, and $a_1^\dagger a_2 + a_2^\dagger a_1$

as the Lindblad dissipation operators:

$$L_S\rho = -\beta\left[\sum_{i=1,2} a_i^\dagger a_i, \left[\sum_{i=1,2} a_i^\dagger a_i, \rho\right]\right] - \phi[a_1^\dagger a_2 + a_2^\dagger a_1, [a_1^\dagger a_2 + a_2^\dagger a_1, \rho]]\,, \quad (4.1)$$

where β and ϕ are the coupling strengths.

The equation of motion for the quantum characteristic function $\chi(\lambda_1, \lambda_2, \lambda_1^*, \lambda_2^*)$ will acquire additional terms from the β-term in L_S:

$$2\beta\left(\lambda_1 \frac{\partial}{\partial\lambda_1} - \lambda_1^* \frac{\partial}{\partial\lambda_1^*}\right)\left(\lambda_2 \frac{\partial}{\partial\lambda_2} - \lambda_2^* \frac{\partial}{\partial\lambda_2^*}\right)\chi - \beta\sum_{i=1,2}\left(\lambda_i \frac{\partial}{\partial\lambda_i} - \lambda_i^* \frac{\partial}{\partial\lambda_i^*}\right)^2 \chi\,. \quad (4.2)$$

On the other hand, the ϕ-term in L_S contributes

$$\begin{aligned}
&-\phi\left[\frac{\partial}{\partial\lambda_2^*}\left(\frac{\partial}{\partial\lambda_1} - \lambda_1^*\right) + \frac{\partial}{\partial\lambda_1^*}\left(\frac{\partial}{\partial\lambda_2} - \lambda_2^*\right)\right]^2 \chi \\
&-\phi\left[\frac{\partial}{\partial\lambda_1}\left(\frac{\partial}{\partial\lambda_2^*} - \lambda_2\right) + \frac{\partial}{\partial\lambda_2}\left(\frac{\partial}{\partial\lambda_1^*} - \lambda_1\right)\right]^2 \chi \\
&+2\phi\left[\frac{\partial}{\partial\lambda_1}\left(\lambda_2 - \frac{\partial}{\partial\lambda_2^*}\right) + \frac{\partial}{\partial\lambda_2}\left(\lambda_1 - \frac{\partial}{\partial\lambda_1^*}\right)\right] \\
&\quad\times\left[\frac{\partial}{\partial\lambda_2^*}\left(\frac{\partial}{\partial\lambda_1} - \lambda_1^*\right) + \frac{\partial}{\partial\lambda_1^*}\left(\frac{\partial}{\partial\lambda_2} - \lambda_2^*\right)\right]\chi\,. \quad (4.3)
\end{aligned}$$

Both the β- and ϕ-terms conserve the total boson number and the system energy, i.e.,

$$\left[\sum_{i=1,2} a_i^\dagger a_i, \hat{H}_{\text{two}}\right] = \left[a_1^\dagger a_2 + a_2^\dagger a_1, \sum_{i=1,2} a_i^\dagger a_i\right] = [a_1^\dagger a_2 + a_2^\dagger a_1, \hat{H}_{\text{two}}] = 0\,. \quad (4.4)$$

Addition of the Lindblad-type dissipation L_S, however, violates the Ehrenfest theorem. Eq. (3.25) ceases to stand in the presence of L_S, and the time derivative of $\langle a_1^\dagger + a_1 + a_2^\dagger + a_2\rangle$ will then depend on $\langle a_1^\dagger + a_1 + a_2^\dagger + a_2\rangle$, which signals the violation of translational invariance of the two-oscillator system. In general, approach to thermal equilibrium is not guaranteed by the Lindblad-type dissipation.

In this section we will focus on the pure dephasing β-term

$$-\beta\left[\sum_{i=1,2} a_i^\dagger a_i, \left[\sum_{i=1,2} a_i^\dagger a_i, \rho\right]\right]\,, \quad (4.5)$$

which, by construction, conserves the total number of the bosons on the two oscillators. It will become apparent that this mechanism of pure-dephasing allows constructions of DFSs which remain unitary during the time evolution. Following the dissipative master equation, the first-order moments obey four coupled

differential equations:

$$\dot{C}_{10,00} = (\mathrm{i}\omega - \beta)C_{10,00} + \mathrm{i}JC_{00,10} \,, \tag{4.6}$$

$$\dot{C}_{01,00} = (-\mathrm{i}\omega - \beta)C_{01,00} - \mathrm{i}JC_{00,01} \,, \tag{4.7}$$

$$\dot{C}_{00,10} = (\mathrm{i}\omega - \beta)C_{00,10} + \mathrm{i}JC_{10,00} \,, \tag{4.8}$$

$$\dot{C}_{00,01} = (-\mathrm{i}\omega - \beta)C_{00,01} - \mathrm{i}JC_{01,00} \,. \tag{4.9}$$

Adding the above four equations confirms the violation of the Ehrenfest theorem for nonzero β. Since the first-order equations are self-contained and homogeneous, for an initial state with zero first-order moments, all first-order moments will remain zero for all times. The second-order moments follow:

$$\dot{C}_{20,00} = 2\mathrm{i}\omega C_{20,00} + \mathrm{i}JC_{10,10} - 4\beta C_{20,00} - \beta(C_{10,00})^2 \,, \tag{4.10}$$

$$\dot{C}_{11,00} = \mathrm{i}J(C_{01,10} - C_{10,01}) + 2\beta C_{10,00}C_{01,00} \,, \tag{4.11}$$

$$\dot{C}_{02,00} = -2\mathrm{i}\omega C_{02,00} - \mathrm{i}JC_{01,01} - 4\beta C_{02,00} - \beta(C_{01,00})^2 \,, \tag{4.12}$$

$$\dot{C}_{10,10} = 2\mathrm{i}\omega C_{10,10} + 2\mathrm{i}J(C_{20,00} + C_{00,20}) - 4\beta C_{10,10} - 2\beta C_{10,00}C_{00,10} \,, \tag{4.13}$$

$$\dot{C}_{10,01} = \mathrm{i}J(C_{00,11} - C_{11,00}) + 2\beta C_{10,00}C_{00,01} \,, \tag{4.14}$$

$$\dot{C}_{01,10} = -\mathrm{i}J(C_{00,11} - C_{11,00}) + 2\beta C_{01,00}C_{10,00} \,, \tag{4.15}$$

$$\dot{C}_{01,01} = -2\mathrm{i}\omega C_{01,01} - 2\mathrm{i}J(C_{02,00} + C_{00,02}) - 4\beta C_{01,01} - 2\beta C_{01,00}C_{00,01} \,, \tag{4.16}$$

$$\dot{C}_{20,00} = 2\mathrm{i}\omega C_{00,20} + \mathrm{i}JC_{10,10} - 4\beta C_{20,00} - \beta(C_{00,10})^2 \,, \tag{4.17}$$

$$\dot{C}_{00,11} = -\mathrm{i}J(C_{01,10} - C_{10,01}) + 2\beta C_{00,10}C_{00,01} \,, \tag{4.18}$$

$$\dot{C}_{00,02} = -2\mathrm{i}\omega C_{00,02} - \mathrm{i}JC_{01,01} - 4\beta C_{00,02}^2 - \beta(C_{00,01})^2 \,. \tag{4.19}$$

Taking into account the first-order equations, the equations for second-order moments can be mapped to equations of operator averages:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_1^\dagger a_1\rangle = -\mathrm{i}J(\langle a_1^\dagger a_2\rangle - \langle a_2^\dagger a_1\rangle) \,, \tag{4.20}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_2^\dagger a_2\rangle = -\frac{\mathrm{d}}{\mathrm{d}t}\langle a_1^\dagger a_1\rangle \,, \tag{4.21}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_1^\dagger a_2\rangle = \mathrm{i}J(\langle a_2^\dagger a_2\rangle - \langle a_1^\dagger a_1\rangle) \,, \tag{4.22}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_2^\dagger a_1\rangle = -\frac{\mathrm{d}}{\mathrm{d}t}\langle a_1^\dagger a_2\rangle \,, \tag{4.23}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_1^\dagger a_2^\dagger\rangle = 2\mathrm{i}\omega\langle a_1^\dagger a_2^\dagger\rangle + \mathrm{i}J(\langle a_1^{\dagger 2}\rangle + \langle a_2^{\dagger 2}\rangle) - 4\beta\langle a_1^\dagger a_2^\dagger\rangle\,, \tag{4.24}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_1 a_2\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\langle a_1^\dagger a_2^\dagger\rangle^*\,, \tag{4.25}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_1^{\dagger 2}\rangle = 2(\mathrm{i}\omega - 2\beta)\langle a_1^{\dagger 2}\rangle + \mathrm{i}2J\langle a_1^\dagger a_2^\dagger\rangle\,, \tag{4.26}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_1^2\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\langle a_1^{\dagger 2}\rangle^*\,, \tag{4.27}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_2^{\dagger 2}\rangle = 2(\mathrm{i}\omega - 2\beta)\langle a_2^{\dagger 2}\rangle + \mathrm{i}2J\langle a_1^\dagger a_2^\dagger\rangle\,, \tag{4.28}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_2^2\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\langle a_2^{\dagger 2}\rangle^*\,. \tag{4.29}$$

Some of the operator averages are not dephased by the β-term:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\langle a_1^\dagger a_1\rangle - \langle a_2^\dagger a_2\rangle) = -4J^2(\langle a_1^\dagger a_1\rangle - \langle a_2^\dagger a_2\rangle)\,, \tag{4.30}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\langle a_1^\dagger a_2\rangle - \langle a_2^\dagger a_1\rangle) = -4J^2(\langle a_1^\dagger a_2\rangle - \langle a_2^\dagger a_1\rangle)\,, \tag{4.31}$$

while others are not modulated by the J-term in $\hat{H}_{\mathrm{two}}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\langle a_2^{\dagger 2}\rangle - \langle a_1^{\dagger 2}\rangle) = 2(\mathrm{i}\omega - 2\beta)(\langle a_2^{\dagger 2}\rangle - \langle a_1^{\dagger 2}\rangle)\,, \tag{4.32}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\langle a_2^2\rangle - \langle a_1^2\rangle) = -2(\mathrm{i}\omega + 2\beta)(\langle a_2^2\rangle - \langle a_1^2\rangle)\,. \tag{4.33}$$

We can construct DFSs with the β-term being the sole dissipative mechanism. The corresponding Hamiltonian for the system plus bath can be written as

$$\hat{H}_{SB} = \hat{H}_{\mathrm{two}} \otimes \hat{I}_{\mathrm{B}} + \hat{I}_S \otimes \hat{H}_{\mathrm{B}} + \hat{F} \otimes \hat{B}\,, \tag{4.34}$$

$$\hat{H}_{\mathrm{B}} = \sum_k \omega_k b_k^\dagger b_k\,, \tag{4.35}$$

$$\hat{F} = \sum_{i=1,2} a_i^\dagger a_i\,, \tag{4.36}$$

where B is the bath operator, I_{B} (I_S) is the identity operator on the physical space of the bath (system or two oscillators), and $\hat{H}_{\mathrm{B}}$ is the bath Hamiltonian. According to Ref. [38], the necessary and sufficient condition for the DFS is that (i) all the basis vectors in the DFS are the degenerate eigenvectors of $\hat{F}$; and (ii) the evolution of the system is confined within the DFS once it starts inside, or in another word, $\hat{H}_{\mathrm{two}}$ leaves the system invariant within the DFS. Any number states of the two oscillators are the eigenstates of $\hat{F}$, and the system Hamiltonian $\hat{H}_{\mathrm{two}}$ commutes with the total occupation number operator or $\hat{F}$. Thus, the number states of the two oscillators with the total occupation number fixed form one of the DFSs. This can be made clear by

introducing the Schwinger boson representation. The two oscillators in $\hat{H}_{\rm two}$ can be viewed as two Schwinger bosons which represent spin operators [39]

$$S^+ = a_1^\dagger a_2 \,, \tag{4.37}$$

$$S^- = a_2^\dagger a_1 \,, \tag{4.38}$$

$$S^z = \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2) \,. \tag{4.39}$$

The spin magnitude S defines a physical subspace within which the dynamics are unperturbed by the global dephasing (the β-term):

$$2S = a_1^\dagger a_1 + a_2^\dagger a_2 \,. \tag{4.40}$$

In the Schwinger boson representation, the system Hamiltonian is written as

$$\hat{H}_{\rm two} = 2\omega S + J(S^+ + S^-) \,. \tag{4.41}$$

The spin raising and lowering operators S^+ and S^- change only the projector of the total spin but not the amplitude. Within the fixed S subspace, the system evolves under the unitary transformation of $\hat{H}_{\rm two}$ only. This is also demonstrated by the first four equations in the set of equations for operator averages, namely, Eqs. (4.20)–(4.23), as well as Eqs. (4.30) and (4.31), where the dissipative β term is not involved in the dynamics. Qubits within the DFS can be used for computation without the risk of being corrupted by noise. We may also construct the DFS for other types of dissipation, for instance, the ϕ dissipative term.

We are also concerned with the separability of the two-oscillator system under the β-term pure-dephasing. We shall employ the number-state representation of partial transposition. The Peres–Horodecki criterion is then invoked to determine its separability. At $t = 0$ the density matrix for the initial the two-mode squeezed state (3.51) has a form [41]

$$\rho(0) = \frac{1}{\cosh^2 s} \sum_{n_1,n_2} (-\tanh s)^{n_1+n_2} |n_1, n_1\rangle\langle n_2, n_2| \,. \tag{4.42}$$

In the Schwinger–boson representation, S can take only integer values (as opposed to half-integers) in states $|n, n\rangle$. Since the projected portions of the two-mode squeezed state onto the subspaces labeled by the integer S only evolve within those same subspaces and remain unaffected by the β-term, only phase coherences between the different S states are dissipated during the process. An expression for the density matrix for arbitrary t can be obtained for the case of $J = 0$ as follows. Assume that in the interaction picture the density matrix read

$$\rho_I(t) = \frac{1}{\cosh^2 s} \sum_{n_1,n_2} \alpha_{n_1 n_2}(t)(-\tanh s)^{n_1+n_2} |n_1, n_1\rangle\langle n_2, n_2| \,. \tag{4.43}$$

The parameters $\alpha_{n_1 n_2}(t)$ are constrained by

$$\dot{\alpha}_{n_1 n_2}(t) = -4\beta(n_1 - n_2)^2 \alpha_{n_1 n_2}(t) \,, \tag{4.44}$$

which follows from

$$\frac{\partial\rho_I}{\mathrm{d}t} = -\beta\left[\sum_{i=1,2} a_i^\dagger a_i, \left[\sum_{i=1,2} a_i^\dagger a_i, \rho_I\right]\right] . \tag{4.45}$$

Therefore,

$$\alpha_{n_1 n_2}(t) = \mathrm{e}^{-4\beta(n_1-n_2)^2 t} . \tag{4.46}$$

The system will asymptotically evolve into

$$\rho_I(\infty) = \frac{1}{\cosh^2 s}\sum_n (\tanh s)^{2n}|n,n\rangle\langle n,n| , \tag{4.47}$$

which is apparently separable from the definition of separability (1.1) and the fact that

$$\sum_{n=0}^{\infty}(\tanh s)^{2n} = \cosh^2 s . \tag{4.48}$$

But for any finite time t, the story is quite different. To see this, one resorts to the Peres–Horodecki criterion, also called the partial transposition criterion, which is a sufficient condition of inseparability in general, and a sufficient and necessary condition for inseparability for the special case of Gaussian wavepackets of two harmonic oscillators. If the bipartite system is composed of subsystems A and B, the partial transposition of a density matrix ρ on the subsystem B is defined as

$$\rho^{\mathrm{T_B}} \equiv \sum_{i,j,k,l}\langle i,k|\rho|j,l\rangle|i,l\rangle\langle j,k| . \tag{4.49}$$

Under partial transpose of the second oscillator (labeled by B),

$$\rho_I^{\mathrm{T_B}} = \frac{1}{\cosh^2 s}\sum_{n_1,n_2}\mathrm{e}^{-4\beta(n_1-n_2)^2 t}(-\tanh s)^{n_1+n_2}|n_1,n_2\rangle\langle n_2,n_1| . \tag{4.50}$$

If the squeezing parameter $s>0$, one finds for a finite t, and $|\psi\rangle = (|0,1\rangle + |1,0\rangle)/\sqrt{2}$

$$\langle\psi|\rho_I^{\mathrm{T_B}}(t)|\psi\rangle = (\langle 0,1|+\langle 1,0|)\frac{\rho_I^{\mathrm{T_B}}}{2}(|0,1\rangle+|1,0\rangle) = -\frac{\mathrm{e}^{-4\beta t}\tanh s}{\cosh^2 s} < 0 . \tag{4.51}$$

Therefore for any finite t the density matrix $\rho(t)$ is inseparable. A similar conclusion can be drawn for $s<0$. At $t=\infty$, however, the argument breaks down because inequality (4.51) no longer stands.

5. Discussions

There is widespread interest in quantum dissipation across many disciplines [18,22,30,42]. Compatibility between quantum mechanics and Markovian motion, which is manifested by the Lindblad exclusion principle of complete positivity, translational invariance, and approach to canonical equilibrium, lies at the center of quantum theories of dissipation. In this paper we study dissipative dynamics of two coupled harmonic oscillators in two types of baths with special attention paid to the separability of the bipartite system and construction of DFSs suitable from quantum computation. The normal-ordered quantum characteristic function is utilized to help probe

the dissipative dynamics of the two oscillator system. In the case of the Agarwal bath exact dynamics are derived with the evaluation of the time-dependent covariance matrix. Separability of the bipartite system is studied with the help of the covariance matrix. For the pure-dephasing Lindblad-type bath equations of motion for first- and second-order moments are derived to elucidate the dynamics of the DFSs labeled by the spin magnitude S in the Schwinger–boson representation. Separability of the uncoupled two-oscillator system ($J=0$) under influence of pure-dephasing is also investigated with the Peres–Horodecki criterion.

While dissipation is well described in classical mechanics by the Langevin or the Fokker–Planck equations, a quantum description of dissipation has been a challenging task. The choice of baths remains at the heart of a successful dissipation theory. The Agarwal bath used in this paper has the appearance of the perfect bath from the physical perspective: it preserves the translational invariance of the system, and delivers the eventual thermal equilibrium to the dissipative process. For most initial conditions and moderate-to-high temperatures density matrix positivity is also supplied by the Agarwal bath. Pathology of the density matrix does appear, and its occurrence is usually accompanied with low temperatures, and will be studied in detail elsewhere [40]. In contrast, the Lindblad-type baths are designed to preserve density matrix positivity at the sacrifice of other desirable traits of the dissipative master equations. If the Lindblad dissipation operator is chosen as the system Hamiltonian $a_1^\dagger a_1 + a_2^\dagger a_2$, dissipation is added to the system dynamics in forms of pure dephasing which does not preserve translational invariance as is evident from adding the first-order equations. Applications of Lindblad-type baths are therefore restricted only to systems other than extended ones. The pathology of dissipative baths is attributed to the failure of a Hamiltonian description of dissipative systems and a lack of proper quantization procedures.

The dynamics of a coupled pair of harmonic oscillators in a dissipative bath have been an important problem in physics and chemistry. An enormous amount of interest has recently emerged in quantum information and computation regarding infinite-dimensional systems, also known as systems of continuous variables, of which a pair of harmonic oscillators is an important paradigm. Our treatment here pays ample attention to the construction and characteristics of dissipative baths in contrast with other work [11–13]. Gaussians states evolving under the Agarwal bath are adequately described by the normal-ordered characteristic functions. Our approach can be readily extended to tripartite systems of harmonic oscillators considering that tri-mode entanglement has come to attention lately with the advent of experiments of continuous variable teleportation [4].

The Peres–Horodecki criterion, also called the partial transposition criterion, has been instrumental in the separability studies of multipartite systems. In this paper we have successfully applied Peres–Horodecki criterion to the bipartite system of twin oscillators. The separability criterion of Duan et al. which utilizes a pair of EPR-type operators is compared with the Peres–Horodecki condition with good agreements. From an initial two-mode squeezed state the system of two oscillators takes less time to become separable at higher temperatures in an Agarwal bath. When the two oscillators are coupled with a direct exchange J-term, the system appears to be able to regain inseparability after spending some time being separable.

In quantum communication maximally entangled states are used for transmitting information between two remote parties. Interactions with the environment inevitably render those states mixed even though they may remain entangled (inseparable). A procedure first proposed by Bennett et al. allows to distill maximally entangled states from the mixed entangled states by using only local operations and classical communication [43]. However, not all mixed entangled states are distillable. Nonpositivity of the partial transposition is also a necessary condition in general for distillibility, and is a necessary and sufficient condition for distillibility in particular for distillibility of 2×2 and 2×3 systems. Gaussian states of two oscillators are either separable or distillable, however, a Gaussian state with PPT has been constructed recently not to be distillable using a subtraction method when there are two oscillators on each of the two subsystems [44]. This calls more detailed investigations of multipartite Gaussian systems.

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Appendix A. One oscillator in an Agarwal bath

For one oscillator in an Agarwal bath, the analytical solutions to the equations of the coefficients C_{mn} for arbitrary initial conditions are given below. The first-order coefficients are

$$C_{10} = \frac{A_1}{\gamma} e^{-(\gamma+\sqrt{\gamma^2-\omega^2})t}(i\omega - \sqrt{\gamma^2-\omega^2}) + \frac{A_2}{\gamma} e^{-(\gamma-\sqrt{\gamma^2-\omega^2})t}(i\omega + \sqrt{\gamma^2-\omega^2}) , \tag{A.1}$$

$$C_{01} = A_1 e^{-(\gamma+\sqrt{\gamma^2-\omega^2})t} + A_2 e^{-(\gamma-\sqrt{\gamma^2-\omega^2})t} . \tag{A.2}$$

The second-order coefficients are

$$C_{11} = \bar{n} + \frac{e^{-2\gamma t}}{\gamma^2-\omega^2}[-B_1\omega^2 + B_1\gamma^2\cosh(2t\sqrt{\gamma^2-\omega^2}) + 2i(B_2-B_3)\gamma\omega\sinh^2(t\sqrt{\gamma^2-\omega^2}) + (B_2+B_3)\gamma\sqrt{\gamma^2-\omega^2}\sinh(2t\sqrt{\gamma^2-\omega^2})] , \tag{A.3}$$

$$C_{02} = \frac{e^{-2\gamma t}}{2(\gamma^2-\omega^2)}[\cosh(2t\sqrt{\gamma^2-\omega^2})(-iB_1\gamma\omega + B_2\gamma^2 + B_3\gamma^2 - 2\omega^2B_3) + \sqrt{\gamma^2-\omega^2}\sinh(2t\sqrt{\gamma^2-\omega^2})(\gamma B_1 - 2i\omega B_3) + iB_1\gamma\omega - \gamma^2(B_2-B_3)] , \tag{A.4}$$

$$C_{20} = \frac{e^{-2\gamma t}}{2(\gamma^2-\omega^2)}\,[\cosh(2t\sqrt{\gamma^2-\omega^2})(iB_1\gamma\omega + B_2\gamma^2 + B_3\gamma^2 - 2\omega^2 B_2)$$
$$+ \sqrt{\gamma^2-\omega^2}\sinh(2t\sqrt{\gamma^2-\omega^2})(\gamma B_1 + 2i\omega B_2)$$
$$- iB_1\gamma\omega + \gamma^2(B_2 - B_3)]\,. \tag{A.5}$$

Here A's and B's are the constants to be determined by the initial conditions. If initially the primary oscillator is in a coherent state with a displacement α [45], i.e.,

$$\rho(t=0) = |\alpha\rangle\langle\alpha|\,, \tag{A.6}$$

then

$$B_1 = -\bar{n}, \quad B_2 = B_3 = 0\,. \tag{A.7}$$

The finite temperature solution for $\chi(\lambda,\lambda^*,t)$ has the form (in the notation of Savage and Walls [46])

$$\chi(\lambda,\lambda^*,t) = \exp[\lambda(u\alpha^* - v\alpha)$$
$$- \lambda^*(u^*\alpha - v\alpha^*) - \bar{n}(\lambda^2 uv + \lambda^{*2}u^*v) - \bar{n}|\lambda|^2(|u|^2 + v^2 - 1)]\,. \tag{A.8}$$

Here u and v are given by

$$u = \frac{e^{-\mu_- t} + e^{-\mu_+ t}}{2} - i\omega\,\frac{e^{-\mu_- t} - e^{-\mu_+ t}}{\mu_- - \mu_+}\,, \tag{A.9}$$

$$v = \gamma\,\frac{e^{-\mu_- t} - e^{-\mu_+ t}}{\mu_- - \mu_+} \tag{A.10}$$

with

$$\mu_\pm = \gamma \pm \sqrt{\gamma^2-\omega^2}\,. \tag{A.11}$$

Appendix B. Characteristic function for the two-oscillator squeeze state

For the two-oscillator squeeze state the normal-ordered characteristic function reads by definition

$$\chi(\lambda_1,\lambda_1^*,\lambda_2,\lambda_2^*,0)$$
$$= \mathrm{Tr}[e^{-s(a_1^\dagger a_2^\dagger - a_1 a_2)}|0\rangle\langle 0|e^{s(a_1^\dagger a_2^\dagger - a_1 a_2)}e^{\lambda_1 a_1^\dagger}e^{-\lambda_1^* a_1}e^{\lambda_2 a_2^\dagger}e^{-\lambda_2^* a_2}]\,. \tag{B.1}$$

We first make use of the identity [41]

$$e^{s(a_1^\dagger a_2^\dagger - a_1 a_2)} = e^{\Gamma a_1^\dagger a_2^\dagger}e^{-\ln(\cosh s)(a_1^\dagger a_1 + a_2^{\dagger a_2 + 1})}e^{-\Gamma a_1 a_2}\,, \tag{B.2}$$

where $\Gamma = \tanh s$. Since $|0\rangle$ is the vacuum state for both oscillators, i.e.,

$$e^{-\ln(\cosh s)(a_1^\dagger a_1 + a_2^\dagger a_2 + 1)}e^{-\Gamma a_1 a_2}|0\rangle = e^{-\ln(\cosh s)}|0\rangle\,, \tag{B.3}$$

one arrives at

$$\chi(\lambda_1,\lambda_1^*,\lambda_2,\lambda_2^*,0)=\mathrm{e}^{-2\ln(\cosh s)}\langle 0|\mathrm{e}^{-\Gamma a_1 a_2}\mathrm{e}^{\lambda_1 a_1^\dagger}\mathrm{e}^{-\lambda_1^* a_1}\mathrm{e}^{\lambda_2 a_2^\dagger}\mathrm{e}^{-\lambda_2^* a_2}\mathrm{e}^{-\Gamma a_1^\dagger a_2^\dagger}|0\rangle\ . \tag{B.4}$$

We would like to move the two exponential operators $\mathrm{e}^{\lambda_1 a_1^\dagger}$ and $\mathrm{e}^{\lambda_2 a_2^\dagger}$ to the left of the double annihilation operator $\mathrm{e}^{-\Gamma a_1 a_2}$. We need to apply to Eq. (B.4) three times the identity

$$\mathrm{e}^A\mathrm{e}^B=\mathrm{e}^{[A,B]}\mathrm{e}^B\mathrm{e}^A \tag{B.5}$$

provided that

$$[[A,B],A]=[[A,B],B]=0\ . \tag{B.6}$$

It then follows from Eq. (B.4) that

$$\chi(\lambda_1,\lambda_1^*,\lambda_2,\lambda_2^*,0)$$

$$=\mathrm{e}^{-2\ln(\cosh s)}\langle 0|\mathrm{e}^{-\Gamma\lambda_1 a_2}\mathrm{e}^{-\Gamma a_1 a_2}\mathrm{e}^{\lambda_2 a_2^\dagger}\mathrm{e}^{-\lambda_1^* a_1}\mathrm{e}^{-\lambda_2^* a_2}\mathrm{e}^{-\Gamma a_1^\dagger a_2^\dagger}|0\rangle \tag{B.7}$$

$$=\mathrm{e}^{-2\ln(\cosh s)}\langle 0|\mathrm{e}^{-\Gamma\lambda_1 a_2}\mathrm{e}^{\lambda_2 a_2^\dagger}\mathrm{e}^{-\Gamma\lambda_2 a_1}\mathrm{e}^{-\Gamma a_1 a_2}\mathrm{e}^{-\lambda_1^* a_1}\mathrm{e}^{-\lambda_2^* a_2}\mathrm{e}^{-\Gamma a_1^\dagger a_2^\dagger}|0\rangle \tag{B.8}$$

$$=\mathrm{e}^{-2\ln(\cosh s)}\langle 0|\mathrm{e}^{-\Gamma\lambda_1\lambda_2}\mathrm{e}^{-\Gamma\lambda_1 a_2}\mathrm{e}^{-\Gamma\lambda_2 a_1}\mathrm{e}^{-\Gamma a_1 a_2}\mathrm{e}^{-\lambda_1^* a_1}\mathrm{e}^{-\lambda_2^* a_2}\mathrm{e}^{-\Gamma a_1^\dagger a_2^\dagger}|0\rangle\ . \tag{B.9}$$

We are now left to evaluate

$$\langle 0|\mathrm{e}^{\alpha^* a_1}\mathrm{e}^{\beta^* a_2}\mathrm{e}^{-\Gamma a_1 a_2}\mathrm{e}^{-\Gamma a_1^\dagger a_2^\dagger}|0\rangle \tag{B.10}$$

with

$$\alpha^*=-\Gamma\lambda_2-\lambda_1^*\ , \tag{B.11}$$

$$\beta^*=-\Gamma\lambda_1-\lambda_2^*\ . \tag{B.12}$$

Expanding all four exponential operators, one has

$$\langle 0|\mathrm{e}^{\alpha^* a_1+\beta^* a_2}\mathrm{e}^{-\Gamma a_1 a_2}\mathrm{e}^{-\Gamma a_1^\dagger a_2^\dagger}|0\rangle$$

$$=\langle 0|\mathrm{e}^{\alpha^* a_1}\mathrm{e}^{\beta^* a_2}\mathrm{e}^{-\Gamma a_1 a_2}\sum_{n=0}^{\infty}(-\Gamma)^n|n,n\rangle \tag{B.13}$$

$$=\langle 0|\mathrm{e}^{\alpha^* a_1}\mathrm{e}^{\beta^* a_2}\sum_{n=0}^{\infty}\sum_{l=0}^{n}\frac{(-\Gamma)^{l+n}n!}{l!(n-l)!}|n-l,n-l\rangle \tag{B.14}$$

$$=\left\langle m,m'\left|\sum_{m=0}^{\infty}\sum_{m'=0}^{\infty}\frac{(\alpha^*)^m(\beta^*)^{m'}}{\sqrt{m!m'!}}\sum_{n=0}^{\infty}\sum_{l=0}^{n}\frac{(-\Gamma)^{l+n}n!}{l!(n-l)!}\right|n-l,n-l\right\rangle \tag{B.15}$$

$$=\sum_{m=0}^{\infty}\sum_{m'=0}^{\infty}\frac{(\alpha^*)^m(\beta^*)^{m'}}{\sqrt{m!m'!}}\sum_{n=0}^{\infty}\sum_{l=0}^{n}\frac{(-\Gamma)^{l+n}n!}{l!(n-l)!}\,\delta_{m,n-l}\delta_{m',n-l} \tag{B.16}$$

$$= \sum_{n=0}^{\infty} (-\Gamma)^{2n} \sum_{l=0}^{n} (-1)^{n-l} \left(\frac{\alpha^* \beta^*}{\Gamma}\right)^{n-l} \frac{n!}{l!(n-l)!(n-l)!} \tag{B.17}$$

$$= \sum_{n=0}^{\infty} L_n\left(\frac{\alpha^* \beta^*}{\Gamma}\right) (\Gamma^2)^n \,, \tag{B.18}$$

where $L_n(x)$ is Laguerre polynomial of order n [47,48]. The series in (B.18) is just the generating function of Laguerre polynomials:

$$g(x,z) = (1-z)^{-1} \mathrm{e}^{-xz/(1-z)} = \sum_{n=0}^{\infty} L_n(x) z^n, \quad |z| < 1 \,. \tag{B.19}$$

Therefore

$$\langle 0 | \mathrm{e}^{\alpha^* a_1 + \beta^* a_2} \mathrm{e}^{-\Gamma a_1 a_2} \mathrm{e}^{-\Gamma a_1^\dagger a_2^\dagger} | 0 \rangle = \frac{1}{1-\Gamma^2} \exp\left(-\frac{\alpha^* \beta^* \Gamma}{1-\Gamma^2}\right) \,, \tag{B.20}$$

where the prefactor $(1-\Gamma^2)^{-1}$ exactly cancels out the prefactor $\mathrm{e}^{-2\ln(\cosh s)}$ in Eq. (B.7). Substituting α^* and β^*, one finally obtains

$$\chi(\lambda_1, \lambda_1^*, \lambda_2, \lambda_2^*, 0)$$

$$= \exp\left(-\Gamma \lambda_1 \lambda_2 - \frac{\alpha^* \beta^* \Gamma}{1-\Gamma^2}\right) \tag{B.21}$$

$$= \exp\left[-\frac{\Gamma}{1-\Gamma^2}(\lambda_1 \lambda_2 + \lambda_1^* \lambda_2^*) - \frac{\Gamma^2}{1-\Gamma^2}(|\lambda_1|^2 + |\lambda_2|^2)\right] \,, \tag{B.22}$$

where the coefficients can be simplified as

$$\frac{\Gamma}{1-\Gamma^2} = \frac{1}{2} \sinh 2s \,, \tag{B.23}$$

$$\frac{\Gamma^2}{1-\Gamma^2} = \sinh^2 s \,. \tag{B.24}$$

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