Multilevel Fast Multipole Algorithm (MLFMA)

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June 2012
Method of Moments

\( \lambda/10 \) triangulation

\[ J(r) = \sum_{n=1}^{N} a_n \, b_n(r) \]

Large numbers of unknowns

\[ \sum_{n=1}^{N} Z_{mn}^E i_n = v_m^E, \quad m = 1, 2, \ldots, N \]

Example (PEC-EFIE):

\[ Z_{mn}^E = \int_{s_m} dr \, t_m(r) \cdot \int_{s_n} dr' \, G(r, r') \cdot b_n(r') \]

Solutions of Matrix-Equations

\[ \sum_{n=1}^{N} Z_{mn}^E a_n = v_m^E, \quad m = 1, 2, \ldots, N \]

Iterative Solutions:

We need acceleration techniques.
MOM: Evaluating the Interactions

**EFIE**

\[ Z_{mn}^E = \int_{S_m} dr_{m}(r) \cdot \int_{S_n} dr' \mathcal{G}(r, r') \cdot b_n(r') \]

**MFIE**

\[ Z_{mn}^M = \int_{S_m} dr_{m}(r) \cdot b_n(r) \]
\[ - \int_{S_m} dr_{m}(r) \cdot \hat{n} \times \int_{S_n} dr' b_n(r') \times \nabla' g(r, r') \]

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**Fast Multipole Method (FMM)**

- **Addition theorem (factorization of Green’s function)**
  \[ g(r, r') = \frac{\exp(ik|D + d|)}{4\pi|D + d|} = \frac{ik}{4\pi} \sum_{l=0}^{\infty} (-1)^l (2l + 1) j_l(kd) h_l^{(1)}(kD) P_l(d \cdot D) \]

  \( d \leq D \)

- **Plane-wave expansion**
  \[ 4\pi i j_l(kd) P_l(d \cdot D) = \int d^3 k \exp(ik \hat{k} \cdot d) P_l(\hat{k} \cdot \hat{D}) \]

- **Diagonalization**
  \[ \frac{\exp(ik|D + d|)}{4\pi|D + d|} \approx \frac{1}{4\pi} \int d^3 \hat{k} \exp(ik \hat{k} \cdot d) T_l(k, \hat{k}, D) \]

- **Translation function**
  \[ T_l(k, \hat{k}, D) = \frac{ik}{4\pi} \sum_{l=0}^{\infty} i^l (2l + 1) h_l^{(1)}(kD) P_l(\hat{k} \cdot \hat{D}) \]
FMM: Evaluating the Interactions

\[ G(r, r') \approx \frac{1}{4\pi} \int d^2 \hat{k} (\mathbf{I} - \hat{k}\hat{k}) e^{ik(r_{r_{fm}} + r_{r_{mc}} + r_{e_{fn}} + r_{f_{mr}})} T_L(k, |r_{ce}|, \hat{r}_{ce} \cdot \hat{k}) \]

**EFIE**

\[ Z_{mn}^E = \frac{1}{4\pi} \int d^2 \hat{k} F_{fmc}^E(\hat{k}) T_L(k, |r_{ce}|, \hat{r}_{ce} \cdot \hat{k}) \cdot F_{fne}^E(\hat{k}) \]

**MFIE**

\[ Z_{mn}^M = \frac{k}{4\pi i} \int d^2 \hat{k} F_{fmc}^M(\hat{k}) T_L(k, |r_{ce}|, \hat{r}_{ce} \cdot \hat{k}) \cdot F_{fne}^M(\hat{k}) \]

**EFIE**

\[ F_{fmc}^E(\hat{k}) = e^{ik(r_{f_{fm}} - r_{f_{mc}})} \int_{S_m} dr e^{ik(r - r_{f_{fm}})} (\mathbf{I} - \hat{k}\hat{k}) \cdot t_m(r) \]

\[ F_{fne}^E(\hat{k}) = e^{ik(r - r_{f_{fn}})} \int_{S_n} dr' e^{-ik(r' - r_{f_{fn}})} (\mathbf{I} - \hat{k}\hat{k}) \cdot b_n(r') \]

**MFIE**

\[ F_{fmc}^M(\hat{k}) = -\hat{k} \times e^{ik(r_{f_{fm}} - r_{f_{mc}})} \int_{S_m} dr e^{ik(r - r_{f_{fm}})} t_m(r) \times \hat{n} \]

\[ F_{fne}^M(\hat{k}) = e^{ik(r - r_{f_{fn}})} \int_{S_n} dr' e^{-ik(r' - r_{f_{fn}})} b_n(r') \]
Fast Multipole Method (Steps)

- Three steps of FMM:
  1. Aggregation:

\[ F_{M}(\hat{k}) = M_{n}(\hat{i}) \cdot b_{n}(r') \]

\[ F_{C'}(\hat{k}) = \int_{S_{n}} dr' \exp(-ik \cdot (r' - r_{C'}))(\bar{I} - \hat{k}\hat{k}) \cdot b_{n}(r') \]

Radiation of the basis function

Radiation of group C'
Fast Multipole Method (FMM)

(2) Translation:

$$F_T^C(\hat{k}) = \sum_{C' \notin \mathcal{N}(C)} T_L(k, |\hat{r}_{ec}', \hat{r}_{ce}', \hat{k}) F_{T'}^{C'}(\hat{k})$$

Incoming wave for group C

Translation function

$$T_L(k, |\hat{r}_{ec}', \hat{r}_{ce}', \hat{k}) = \frac{ik}{4\pi} \sum_{l=0}^{L} (2l + 1)j_l(k |\hat{r}_{ec}') P_l(\hat{r}_{ce}' \cdot \hat{k})$$

Error Source: Truncation of an infinite summation

(3) Disaggregation and reception:

$$\sum_{n=1}^{N} Z_m a_n = \frac{1}{4\pi} \int d^2\hat{k} F_{mn}(\hat{k}) \cdot F_T^C(\hat{k})$$

Receiving pattern of the testing function

$$F_{mn}(\hat{k}) = -\hat{k} \times \int_{S_m} d\hat{r} \exp(ik \cdot (\hat{r} - \hat{r}_c)) t_m(\hat{r}) \times \hat{n}$$

Error Source: Angular integration over unit sphere
FMM Considerations

1) Truncation of infinite summation:

\[ T_L(k, D, \theta) = \frac{ik}{4\pi} \sum_{l=0}^{L} i^l (2l + 1) h_t^{(1)}(kD) P_l(\cos \theta) \]

\[ L \approx kd + 1.8d_0^{2/3}(kd)^{1/3} \]

2) Angular integration:

\[ Z_{mn}^M = \frac{k}{4\pi i} \int d^2 \hat{k} F_{fmc}^M(\hat{k}) T_L(k, |r_{cc'}|, \hat{r}_{cc'} \cdot \hat{k}) \cdot F_{fmc}^M(\hat{k}) \]

\[ K=2L^2 \]
Fast Multipole Method

\[ \sum_{n=1}^{N} Z_{mn} a_n = \sum_{n \in N(G)} \sum_{m \in G} Z_{mn} a_n \]

Near-field interactions

\[ + \frac{1}{4\pi} \int d^2k \ F_{mG}(\hat{k}) \cdot \sum_{n \in G} T_L(k, r_{G'}) \sum_{n \in G} F_{nG'}(\hat{k}) a_n \]

Far-field interactions

\[ m = 1, 2, \ldots, N \]

- Aggregation
- Disaggregation
- Translation
- Basis functions
- Testing function
- Incoming field
- Radiation pattern

Cluster of basis functions

Cluster of testing functions
Multilevel Fast Multipole Algorithm
Recursive clustering

Three-dimensional object

Largest cubic box
Multilevel Fast Multipole Algorithm

- Recursive clustering
- Tree structure

MLFMA Tree Structure

Level: 3
Level: 2
Unknown Level
Recursive Clustering

• Tree structure:

Multilevel Fast Multipole Algorithm

• Tree structure:

• Complexity: $O(N \log N)$
Multilevel Fast Multipole Algorithm

- Consider aggregation on the entire tree-structure.

- Consider translation on the entire tree-structure.
MLFMA Interactions

Acceleration with MLFMA

• Solve

\[ \sum_{n=1}^{N} Z_{mn}^E a_n = v_m^E \]

Matrix-Vector Multiplication
\[ \bar{Z} \cdot x = y \]

Iterative Algorithm

• Processing time for a matrix-vector product:

MOM [O(N^3)] \rightarrow FMM [O(N^{3/2})] \rightarrow MLFMA [O(N \log N)]

• Memory requirement:

MOM [O(N^2)] \rightarrow FMM [O(N^{3/2})] \rightarrow MLFMA [O(N \log N)]
Multilevel FMM

• Apply the FMM concept in multi-level scheme:
  Group the groups!

• Form tree structure:

```
  Parent Clusters
  Level i

  Clusters
  Level i+1
```

Radiation of the parent clusters
Interpolation

```
Radiation of the clusters
```

Aggregation between the levels
Multilevel FMM

Aggregation:

\[ F_{C_i}^{(k)}(\hat{k}) = \sum_{c_{i+1} \in C_{i+1}} \beta_{c_{i+1}} \cdot \frac{F_{C_{i+1}}^{(k)}}{I_{i+1}} \]

where

\[ \beta_{c_{i+1}} = \exp \left[ i \cdot (r_{c_{i}} - r_{c_{i+1}}) \right] \]

Translation: Required in each level.

\[ F_{T_i}^{(k)}(\hat{k}) = \sum_{c_i \in C_i} T_{L_i}(k_i, r_i, \hat{r}_i, k_i) \cdot F_{C_i}^{(k)}(\hat{k}) \]

Use symmetry for efficiency:

\[ \tau(l) \approx 1.73ka_l + 2.16(d_0)^{2/3}(ka_l)^{1/3} \]
Multilevel FMM

Disaggregation between the levels

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Disaggregation:

\[
F_{TD}^{C_{i+1}}(k) = F_{TD}^{C_i}(k) + T_{i+1,i} \cdot \beta_{C_{i+1}C_{i}} c_{i} + P_{TD}^{C_{i+1}}(k)
\]

Translation Disaggregation
Interpolation and Anterpolation

Lagrange Interpolation

Lagrange interpolation employing $4 \times 4$ points (shaded circles) located in the coarse grid to evaluate the function at a point (star) located in the fine grid. Sampling values of $\theta$ and $\varphi$ are specified in radians and selected from a practical case.
Multilevel FMM

- Between the levels interpolation and anterpolation algorithms are required:

Error Source: Interpolation between levels

- Processing time for a matrix-vector product:
  MOM $[O(N^2)] \rightarrow$ FMM $[O(N^{3/2})] \rightarrow$ MLFMA $[O(N\log N)]$

- Memory requirement:
  MOM $[O(N^2)] \rightarrow$ FMM $[O(N^{3/2})] \rightarrow$ MLFMA $[O(N\log N)]$

Multilevel Fast Multipole Algorithm

- Aggregation:
  - Performed from bottom to top
  - Local interpolations are used
  - $O(N)$ operations per level

- Translation:
  - $O(1)$ testing clusters for each basis cluster
  - $O(N)$ operations per level
  - $O(1)$ different translation operators per level (for cubic clusters)
### Complexity of MLFMA

**Major Parts of MLFMA and Their Computational Requirements**

<table>
<thead>
<tr>
<th>Part</th>
<th>Memory</th>
<th>Complexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>MVM</td>
<td>( \sum_{l=1}^{L} N_l [\tau(l) + 1]^2 )</td>
<td>( O(N \log N) )</td>
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<tr>
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<tr>
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<tr>
<td>Near-Field Interactions</td>
<td>( N^2 / N_1 )</td>
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**Processing Time**

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Note: \( \alpha_l \) and \( d_l \) represent relative weights for levels \( l = 1, 2, \ldots, L \).

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### Multilevel Fast Multipole Algorithm

#### SETUP

- **Calculation of Nearfield Interactions**
- **Calculation of Radiation & Receiving Patterns**
- **Calculation of Translation Operators**

#### ITERATIVE SOLUTION

- **Matrix-vector multiplications**
- **Using Nearfield Interactions**
- **Aggregation & Disaggregation**
- **Translation**